Mathematics 371 Fall, 2012 (SHATZ) ①

Assignments #6 (Last), due Dec 11, 2012

Part A

AI) 355/4.4 and 356/5.5

AII) 356/5.6

AIII) Read Props 12.2.19 & 12.2.20 & prove: If $f(X) \in \mathbb{R}[X]$, and if $f(X)$ is irred., then $\deg(f) \le 2$. Now say $f(X)$ has deg 2 & is irred. Prove that the two fields $(\mathbb{C} =)\ \mathbb{R}[X]/(X^2+1)$ and $\mathbb{R}[X]/(f(X))$ are isomorphic.

AIV) 380/4.9 & 4.12

AV) 381/5.4 & 5.5

Part B

BI) a) Mimic our proof that $\mathbb{Z}/n\mathbb{Z} \cong \prod_{i=1}^{t} \mathbb{Z}/p_i^{e_i}$

10 to show: If R is a PID and I is an ideal of R, then $R/I \cong$ finite prod of rings $R/(\pi^e)$, where π is a prime elt of R and $e > 0$.

5 b) Say I is an ideal of the PID, R. Show $\exists$ only finitely many prime ideals, $\mathfrak{q}$, of R so that $\mathfrak{q} \supseteq I$, provided $I \neq (0)$.

c) Suppose b) holds without the proviso that

10 $I \neq (0)$. Then R has but finitely many prime ideals, call them $P_1, P_2, \ldots, P_t$. Let $J = \bigcap_{j=1}^{t} P_j$. If $x \in J$, show that $x^N = 0$ for some N.

(Hint: Let $S = \{x^n \mid n \ge 0\}$, try to form $S^{-1}R$.)

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BII) Let $X_{11}, X_{12}, \ldots, X_{1n}, X_{2n}, \ldots, X_{n1}, \ldots, X_{nn}$ be n^2 indep. variables, write R for the UFD

$$R = \mathbb{Z}[X_{11}, \ldots, X_{1n}, \ldots, \ldots, X_{nn}].$$

Notice that R is what we get from the entries of the "general matrix"

$$\mathfrak{X} = \begin{pmatrix} X_{11} & \cdots & X_{1n} \\ X_{21} & \cdots & X_{2n} \\ \vdots & & \vdots \\ X_{n1} & \cdots & X_{nn} \end{pmatrix}.$$

10 a) Let $n = 2$ and consider $\det \mathfrak{X} = X_{11}X_{22} - X_{21}X_{12}$. Prove that $\det \mathfrak{X}$ is irreducible in R.

b) Remember that the hallmark of a poly ring $\mathbb{Z}[Y_1, \ldots, Y_t]$ is that all homomorphisms from it to

20 any comm. ring S are given by assigning elts $\sigma_1, \ldots, \sigma_t$ of S to the variables $Y_1, \ldots, Y_t$ and evaluating each poly $f(Y_1, \ldots, Y_t)$ on $\sigma_1, \ldots, \sigma_t$ to get an elt of S. Use this to prove:

For any n, $\det \mathfrak{X}$ is an irred. poly in R.

(Hint: Try $n = 3$ first and substitute some values of $\mathbb{Z}$ for some of the 9 variables $X_{11}, \ldots, X_{33}$. Try to use a).)

BIII)

8 a) 382/M.3

8 b) 382/M.7

4 c) 382/M.10

B IV) It's known that if p is prime, a group of order p^2 is abelian. Let G be a non-abelian group, then there are conjugacy classes in G with more than one element. Generally, the number of conjugacy classes goes to ∞ as $\#(G) \to \infty$. So look at the ratio $\frac{c(G)}{\#(G)}$, where $c(G) =$ number of distinct conj. classes of G.

10 a) Prove $\frac{c(G)}{\#(G)} \leq 5/8$.

5 b) Is there a finite group, G, where $\frac{c(G)}{\#(G)} = \frac{5}{8}$?

c) Now say p is the smallest prime to divide $\#(G)$. Show for such non-abelian G that

10 $$\frac{c(G)}{\#(G)} \leq \frac{1}{p} + \frac{1}{p^2} - \frac{1}{p^3}$$

Again, is there a group where we get equality? Try a guess based on b).