Mathematics 371 Oct 2, 2012 (Shatz)

Assignment #2 Due October 16, 2012

Part A (From Artin)

AI) 354/1.3

AII) 354/1.8 and 2.1

AIII) 355/3.7

AIV) 355/3.10

AV) 356/6.1 and 6.2

AVI) 357/7.1, 7.2, 7.3

Part B (To be turned in.)

BI) Here we study integer solutions to some polynomial equations. The linear case is easy, the quadratic case more interesting. We look at

$$X^2+Y^2 = kZ^2 \quad (k \text{ given}, \in \mathbb{Z})$$

Of course, $X=Y=Z=0$ is a sol'n, this is the trivial solution — we ignore it being of no interest.

a) For $k=1$, $X^2+Y^2=Z^2$ was already discussed by the Greek mathematicians of the 4th century B.C. We demand that $XYZ \neq 0$ (else everything is trivial). Prove that all solutions (in integers) to $X^2+Y^2=Z^2$ are given by

$$X = \tfrac{1}{2}(a^2-b^2)c$$

$$Y = abc$$

$$Z = \tfrac{1}{2}(a^2+b^2)c$$

where a, b, c are integers all odd or all even. (That these X, Y, Z are sol'ns is simple high school algebra. You must show all integer solutions arise from these formulae (simply) by varying a, b, c in $\mathbb{Z}$ making sure all are odd or all are even.)

b) Now look at $X^2+Y^2=2Z^2$. Give an analogous set of formulae for X, Y, Z from integers a, b, c (all odd or all even) which give rise to all solutions.

c) Can you proceed the same way for $X^2+Y^2=3Z^2$? Are there any solutions? Prove all statements you make.

d) Repeat question c) for $X^2+Y^2=4Z^2$; for $X^2+Y^2=5Z^2$; for $X^2+Y^2=7Z^2$; for $X^2+Y^2=11Z^2$; for $X^2+Y^2=13Z^2$.

BII) We are concerned with the notions of finite and infinite sets. Put aside all your preconceived notions and just use logic and the following definitions (you can't use $\mathbb{Z}$ until it appears later in the problem):

A set T ~~(not empty)~~ is infinite $\Longleftrightarrow$ there is a proper subset, S, of T and a bijection $\varphi: S \to T$. (S proper means $S \neq \emptyset$ and $S \neq T$.)

A set T is finite $\Longleftrightarrow$ it is not infinite (that is, there is no proper subset, S, with a bijection $\varphi: S \to T$).

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