6. Find the arc length of the curve $y = \frac{1}{3}x^{\frac{3}{2}}$ between the points $(0,0)$ and $(1,\frac{1}{3})$.

(10 points)

① $f(x) = \frac{1}{3}x^{\frac{3}{2}}, \quad f'(x) = \frac{1}{2}x^{\frac{1}{2}}$

$\sqrt{1+(f'(x))^2} = \sqrt{1+(\frac{1}{2}x^{\frac{1}{2}})^2} = \sqrt{1+\frac{1}{4}x}$

x is the variable

$a=0, \quad b=1$

② $L = \int_a^b \sqrt{1+(f'(x))^2}\,dx$

$= \int_0^1 \sqrt{1+\frac{1}{4}x}\,dx$

③ $\int \sqrt{1+\frac{1}{4}x}\,dx$

$u = 1+\frac{1}{4}x \qquad du = \frac{1}{4}dx \qquad dx = 4du$

$\int \sqrt{u}\;4du = 4\cdot\frac{2}{3}u^{\frac{3}{2}}+C$

$= \frac{8}{3}u^{\frac{3}{2}}+C$

$= \frac{8}{3}(1+\frac{1}{4}x)^{\frac{3}{2}}+C$

④ $L = \int_0^1 \sqrt{1+\frac{1}{4}x}\,dx = \frac{8}{3}(1+\frac{1}{4}x)^{\frac{3}{2}}\Big|_0^1$

$= \frac{8}{3}\left[(1+\frac{1}{4})^{\frac{3}{2}} - 1^{\frac{3}{2}}\right]$

$= \boxed{\frac{8}{3}\left[(\frac{5}{4})^{\frac{3}{2}} - 1\right]}$