MATH 644, FALL 2011, HOMEWORK 8

Exercise 1. *(The adjoint of $S(t)$) [5 points]*

Let $S(t)$ denote the linear Schrödinger propagator defined in class. Show that:

$$(S(t))^* = S(-t)$$

where $\cdot^*$ denotes the adjoint with respect to the $L^2(\mathbb{R}^n)$ inner product.

Exercise 2. *(Solving the wave equation by using the Fourier transform) [10 points]*

a) Suppose that $f, g \in \mathcal{S}(\mathbb{R}^n)$. Consider the initial value problem:

$$\begin{cases} (\frac{\partial^2}{\partial t^2} - \Delta_x)u = 0 \text{ on } \mathbb{R}^n_x \times \mathbb{R}_t \\ u = f, u_t = g \text{ on } \mathbb{R}^n \times \{t = 0\}. \end{cases} \tag{1}$$

Find $\widehat{u}(\xi, t)$, where $\widehat{\cdot}$ denotes the Fourier transform in the x variable.

b) Use the Fourier transform and part a) to show that:

$$\|\nabla u(x,t)\|^2_{L^2(\mathbb{R}^n_x)} + \|\frac{\partial}{\partial t}u(x,t)\|^2_{L^2(\mathbb{R}^n_x)} = \|\nabla f(x)\|^2_{L^2(\mathbb{R}^n_x)} + \|g(x)\|^2_{L^2(\mathbb{R}^n_x)}.$$

This method gives us an alternative derivation of the conservation of energy for the wave equation.

Exercise 3. *(Localization of functions in the frequency space) [15 points]*

a) Given $N > 1$, and $\psi \in C^\infty(\mathbb{R}^d)$ which is radial and satisfies:

$$\begin{cases} \psi = 1 \text{ on } \frac{3}{4} \leq |\xi| \leq \frac{5}{4} \\ \psi = 0 \text{ on } |\xi| \leq \frac{1}{2} \text{ and on } |\xi| \geq 2. \end{cases} \tag{2}$$

We note that ψ defined in this way is a smooth approximation of the characteristic function of the annulus $|\xi| \sim 1$. For the ψ defined in (2), we define an operator P_N on $L^2(\mathbb{R}^d)$ by:

$$(P_N f)\widehat{\ }(\xi) := \psi(\frac{\xi}{N})\widehat{f}(\xi).$$

a) Explain why it is clear by construction that:

$$\|P_N f\|_{L^2(\mathbb{R}^d)} \leq \|f\|_{L^2(\mathbb{R}^d)}.$$

b) Express P_N as a convolution operator, i.e. find K_N such that:

$$P_N f = K_N * f.$$

c) Using part b), prove that for $1 \leq p \leq \infty$, one has:

$$\|P_N f\|_{L^p(\mathbb{R}^d)} \leq C\|f\|_{L^p(\mathbb{R}^d)}.$$

(Strictly speaking, we are defining K_N on $L^2 \cap L^p$ and we are extending the definition by density.)

[HINT: Use Young's Inequality.]

d) Show more generally that for all $1 \leq p \leq q \leq \infty$, one has:

$$\|P_N f\|_{L^q(\mathbb{R}^d)} \leq CN^{\frac{d}{p} - \frac{d}{q}}\|f\|_{L^p(\mathbb{R}^d)}.$$

e) Show that the result in d) can be improved to

$$\|P_N f\|_{L^q(\mathbb{R}^d)} \leq CN^{\frac{d}{p} - \frac{d}{q}}\|P_N f\|_{L^p(\mathbb{R}^d)}.$$ *(See next page).*

[HINT: Write $K_N f = \tilde{K}_N K_N f$ *where* $\tilde{K}_N$ *is an operator of a similar type as* K_N *which localizes to a slightly larger region in the frequency space.]*

This homework assignment is due in class on Friday, December 9. Good Luck!