MATH 644, FALL 2011, HOMEWORK 7

Exercise 1. *(An interpolation inequality) [5 points]*

Suppose that $1 \leq p, q \leq \infty$ and suppose that $\theta \in [0,1]$ is given. Define r by:

$$\frac{1}{r} := \frac{\theta}{p} + \frac{1-\theta}{q}$$

Show that:

$$\|f\|_{L^r} \leq \|f\|_{L^p}^{\theta} \|f\|_{L^q}^{1-\theta}.$$

Exercise 2. *(More on the Hausdorff-Young Inequality) [15 points]*

a) Suppose that $1 \leq p, q \leq \infty$ are such that there exists $C > 0$ with the property that:

$$\|\widehat{f}\|_{L^q_x} \leq C \|f\|_{L^p_x}$$

for all $f \in L^p(\mathbb{R}^n)$. Show that $\frac{1}{p} + \frac{1}{q} = 1$. (HINT: Use scaling).

b) Suppose that $1 \leq p \leq \infty$ is such that there exists $C > 0$ with the property that:

$$\|\widehat{f}\|_{L^{p'}_x} \leq C \|f\|_{L^p_x}$$

for all $f \in L^p(\mathbb{R}^n)$. Here p' denotes the Hölder conjugate of p, i.e. $\frac{1}{p} + \frac{1}{p'} = 1$. Show that necessarily $1 \leq p \leq 2$.

HINT: This is a subtle construction. The idea is that, given $N \in \mathbb{N}$, one defines the function: $f_N(x) := \sum_{n=1}^{N} e^{2\pi i x \cdot (nv)} g(x - nv)$ for g a Gaussian and $v \in \mathbb{R}^n$, a unit vector.

i) How is the Fourier transform of f_N related to f_N?

ii) What is a good lower bound for $\|f_N\|_{L^p}$? (It is good to look at parts of f_N near xv.)

iii) What is an upper bound for $\|f_N\|_{L^1}$ and for $\|f_N\|_{L^\infty}$?

iv) Use iii) and Exercise 1 to deduce an upper bound for $\|f_N\|_{L^p}$.

v) Is it possible to deduce that $\|f_N\|_{L^p} \sim N^r$ for some power r? How about $\|\widehat{f_N}\|_{L^{p'}}$?

Exercise 3. *(Generalized Young's Inequality) [5 points]*

Suppose that $1 \leq p, q, r \leq \infty$ are such that $\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1$. By using the Riesz-Thorin Interpolation Theorem, show that:

$$\|f * g\|_{L^r} \leq \|f\|_{L^p} \cdot \|g\|_{L^q}.$$

Exercise 4. *(A refinement of Young's inequality when $r = \infty$) [5 points]*

Suppose that $1 < p < \infty$ and suppose that $f \in L^p(\mathbb{R}^n)$ and $g \in L^{p'}(\mathbb{R}^n)$. Show that $f * g$ is uniformly continuous and that it decays to zero at infinity.

HINT: Recall that $\lim_{t \to 0} \|f(\cdot + t) - f\|_{L^p} = 0$.

This homework assignment is due in class on Wednesday, November 30. Good Luck!