MATH 644, FALL 2011, HOMEWORK 6

Exercise 1. *(Reflection of traveling waves) [5 points] Solve the equation:*

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 \;\; on \; (0,L) \times (0,+\infty) \\ u = g, u_t = 0 \;\; on \; (0,L) \times \{t=0\} \\ u = 0 \;\; on \; \big(\{0\} \times (0,+\infty)\big) \cup \big(\{L\} \times (0,+\infty)\big). \end{cases} \tag{1}$$

by converting it to a problem on $\mathbb{R}$.

Exercise 2. *(A derivation of d'Alembert's formula by using a change of variables) [Evans (1st edition), Problem 15 in Chapter 2; 5 points]*

a) Show that the general solution of the PDE $u_{xy} = 0$ *is given by:*

$$u(x,y) = F(x) + G(y)$$

for differentiable functions F *and* G.

b) Using the change of variables $\xi = x + t, \eta = x - t$, *show that* $u_{tt} - u_{xx} = 0$ *if and only if* $u_{\xi\eta} = 0$.

c) Use a) and b) to rederive d'Alembert's formula.

Exercise 3. *(Equipartition of energy) [Evans, Problem 17 in Chapter 2; 10 points] Let* $u \in C^2(\mathbb{R} \times [0,+\infty))$ *solve the initial value problem for the wave equation in one dimension:*

$$\begin{cases} u_{tt} - u_{xx} = 0 \;\; on \; \mathbb{R} \times (0,+\infty) \\ u = g, u_t = h \;\; on \; \mathbb{R} \times \{t=0\}. \end{cases} \tag{2}$$

Suppose that $g \in C^2(\mathbb{R})$, $h \in C^1(\mathbb{R}$ *both have compact support. The* ***kinetic energy*** *is defined by:*

$$k(t) := \frac{1}{2} \int_{-\infty}^{+\infty} u_t^2(x,t)dx$$

and the ***potential energy*** *is defined by:*

$$p(t) := \frac{1}{2} \int_{-\infty}^{+\infty} u_x^2(x,t)dx.$$

a) Show that $k(t) + p(t)$ *is constant in time by using d' Alembert's formula. Hence, the total energy is conserved in time. We saw how to prove this fact in class by using the equation.*

b) Moreover, show that $k(t) = p(t)$ *for sufficiently large* t. *In other words, the total energy gets equally partitioned into the kinetic and potential part over a sufficiently long time.*

Exercise 4. *(A nonlinear problem related to the wave equation) [10 points]*

Consider the nonlinear PDE:

$$\begin{cases} u_{tt} - \Delta u + u_t^2 - |\nabla u|^2 = 0 \;\; on \; \mathbb{R}^3 \times (0,+\infty) \\ u = 0, u_t = g \;\; on \; \mathbb{R}^3 \times \{t=0\}. \end{cases} \tag{3}$$

Let us suppose that $g \in C^\infty(\mathbb{R}^3)$ *and that* g *has compact support.*

a) Show that $v := e^u$ *solves:*

$$\begin{cases} v_{tt} - \Delta v = 0 \;\; on \; \mathbb{R}^3 \times (0,+\infty) \\ v = 1, v_t = g \;\; on \; \mathbb{R}^3 \times \{t=0\}. \end{cases} \tag{4}$$

b) Use Kirchoff's formula to solve for v.

We want to show that for g "sufficiently small", v is positive, and hence $u = ln(v)$ is well-defined and is a global smooth solution of (3).

c) Observe that this reduces to showing that: $t⨍_{\partial B(x,t)} g(y)dS(y)$ is uniformly small in t, x.

d) By fixing $x \in \mathbb{R}^3$ and rescaling, note that one has to consider $t⨍_{\partial B(0,1)} g(x+t\xi)dS(\xi)$.

e) Use the fundamental theorem of Calculus in t to write $g(x+t\xi) = -\int_t^{+\infty} \frac{\partial}{\partial s}[g(x+s\xi)]ds$. One is then reduced with estimating an integral in s and ξ. Estimate this integral by using polar coordinates centered at x. Deduce that:

$$\big|t⨍_{\partial B(0,1)} g(x+t\xi)dS(\xi)\big| \leq \frac{C}{t}\|\nabla g\|_{L^1}$$

This bound is good for large t.

f) How should we argue for small t?

Exercise 5. *(A decay estimate for the wave equation) [Evans (1st edition), Problem 18 in Chapter 2; 10 points]*

Let u solve:

$$\begin{cases} u_{tt} - \Delta u = 0 \text{ on } \mathbb{R}^3 \times (0,+\infty) \\ u = g, u_t = h \text{ on } \mathbb{R}^3 \times \{t=0\}. \end{cases} \tag{5}$$

Suppose that $g, h \in C^\infty(\mathbb{R}^3)$ have compact support. Show that there exists a constant $C > 0$ such that:

$$|u(x,t)| \leq \frac{C}{t}$$

for all $x \in \mathbb{R}^3$ and $t > 0$. (HINT: Use Kirchoff's formula. The bound is not immediately obvious from the formula. One can argue as in the previous problem by using the fundamental theorem of Calculus and polar coordinates centered at a fixed $x \in \mathbb{R}^3$.)

This homework assignment is due in class on Wednesday, November 16. Good Luck!