MATH 644, FALL 2011, HOMEWORK 5

Exercise 1. *(An alternative derivation of the heat kernel in one dimension) [Evans, Problem 11 in Chapter 2; 5 points]*

Assume $n = 1$ *and* $u = v(\frac{x^2}{t})$.

a) Show that $u_t = u_{xx}$ *if and only if:*

$$4zv''(z) + (2+z)v'(z) = 0, \mathit{for}\, z > 0. \tag{1}$$

b) Show that the general solution to (1) *is given by:*

$$v(z) = c\int_0^z e^{-\frac{s}{4}} s^{-\frac{1}{2}} ds + d.$$

c) Differentiate $v(\frac{x^2}{t})$ *with respect to* x *and select the constant* c *properly in order to obtain the fundamental solution* Φ *for* $n = 1$.

Exercise 2. *(An explicit solution to a heat-type equation) [Evans, Problem 12 in Chapter 2; 5 points]*

Find an explicit solution (in terms of Φ*) of:*

$$\begin{cases} u_t - \Delta u + cu = f \text{ on } \mathbb{R}^n \times (0, +\infty) \\ u = g \text{ on } \mathbb{R}^n \times \{t = 0\}. \end{cases} \tag{2}$$

Here, $f \in C(\mathbb{R}^n \times (0, +\infty))$ *and* $g \in C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$.

Exercise 3. *(A maximum principle and a method for deducing decay estimates) [Taken in part from Problem 14 in Chapter 2 of Evans; 10 points]*

a) We say that $v \in C_1^2(U_T)$ *is a subsolution of the heat equation if:*

$$v_t - \Delta v \leq 0 \text{ on } U_T.$$

Explain briefly how the proof of the mean value formula for the heat equation can be modified to show that, for a subsolution v*, one has:*

$$v(x,t) \leq \frac{1}{C_n r^n} \int\int_{E(x,t;r)} v(y,s)\frac{|x-y|^2}{(t-s)^2} dyds.$$

for all $E(x,t;r) \subseteq U_T$. *(with notation as in the proof of the mean value theorem) Deduce that:*

$$\max_{\overline{U_T}} v = \max_{\Gamma_T} v.$$

b) Suppose that $U = (0,1)$ *and* $u \in C_1^2(U_T)$ *solves:*

$$\begin{cases} u_t(x,t) - u_{xx}(x,t) = 0 \text{ for } (x,t) \in (0,1) \times (0, +\infty) \\ u(x,0) = x(1-x) \text{ for } x \in [0,1] \\ u(0,t) = u(1,t) = 0 \text{ for } t > 0. \end{cases} \tag{3}$$

Show that $u \geq 0$.

c) For u *as in part b), show that there exist constants* $\alpha, \beta > 0$ *such that:*

$$u(x,t) \leq \alpha x(1-x)e^{-\beta t}.$$

Deduce that $u(x,t) \to 0$ *as* $t \to +\infty$.

This homework assignment is due in class on Monday, October 24. Good Luck!