# MATH 644, FALL 2011, HOMEWORK 4

**Exercise 1.** *(An explicit form of Harnack's inequality; Evans, Problem 6 from Chapter 2)*
*a) Use Poisson's formula for the ball to prove the inequality:*

$$r^{n-2}\frac{r-|x|}{(r+|x|)^{n-1}}u(0) \leq u(x) \leq r^{n-2}\frac{r+|x|}{(r-|x|)^{n-1}}u(0)$$

*whenever $u$ is a non-negative harmonic function on $B(0,r) \subseteq \mathbb{R}^n$.*
*b) Use part a) to deduce that there exists a constant $C_n > 0$, depending only on $n$ such that, whenever $u$ is a non-negative harmonic function on $B(a,R) \subseteq \mathbb{R}^n$, one has:*

$$\max_{B(a,\frac{R}{2})} u \leq C_n \min_{B(a,\frac{R}{2})} u.$$

**Exercise 2.** *(A condition for convergence of a sum of non-negative harmonic functions)*
*Suppose that $U \subseteq \mathbb{R}^n$ is open and connected and suppose that $(u_n)$ is a sequence of non-negative harmonic functions on $U$ such that the series $\sum_n u_n$ converges uniformly at $x_0 \in U$.*

*a) Show that the series $\sum_n u_n$ converges uniformly on any compact subset $K \subseteq U$.*
*[HINT: Argue as in the proof of Harnack's inequality and cover $K$ by a chain of finitely many balls with non-empty intersection starting from $x_0$. Part b) of the previous exercise is useful too.]*

*b) Deduce that the sum $\sum_n u_n$ is harmonic on U.*

**Exercise 3.** *(A reflection principle for the Laplace equation; Evans, Problem 9 from Chapter 2)*
*Let $U^+ := \{x \in \mathbb{R}^n;\ |x| < 1,\ x_n > 0\}$ be an open half-ball. a)We assume that $u \in C^2(U^+)$ is harmonic on $U^+$ and $u = 0$ on $\partial U^+ \cap \{x_n = 0\}$. For $x \in B(0,1)$, we let:*

$$v(x) := \begin{cases} u(x), \text{ if } x_n \geq 0 \\ -u(x_1, ..., x_{n?1}, -x_n), \text{ if } x_n < 0. \end{cases} \tag{1}$$

*Show that $v$ is harmonic on $U$.*

*b) Assume now only that $u \in C^2(U^+) \cap C^0(U^+)$, and prove the analogous statement. [Remark: This is not immediately obvious since we cant immediately take derivatives in $x_n$ near $x_n = 0$. One should use Poisson's formula for the ball by which we can solve the Laplace equation given sufficiently regular boundary data. It is also useful to know that we can deduce symmetry properties of the solution to Laplace's equation from symmetry properties of the boundary data.]*

**Exercise 4.** *(Removable singularities for the Laplace equation on $\mathbb{R}^2$) Suppose that $U \subseteq \mathbb{R}^2$ is open and bounded and suppose that for some $x_0 \in U$, we are given a function $u \in C^2(U \setminus \{x_0\})$ which satisfies:*

$$\Delta u(x) = 0, \text{ and } |u(x)| \leq M, \text{ for some constant } M > 0 \text{ and for all } x \in U \setminus \{x_0\}.$$

*Show that there exists $\tilde{u} \in C^2(U)$, which is harmonic on $U$ and agrees with $u$ on $U \setminus \{x_0\}$.*
*[HINT: It is good to first reduce to the case $x_0 = 0$ and $U = B(0,R)$. By Poisson's formula for $B(0,R)$, we can solve:*

$$\begin{cases} \Delta v = 0 \text{ on } B(0,R) \\ v = u \text{ on } \partial B(0,R). \end{cases} \tag{2}$$

*We want to show that* $w := u - v = 0$ *on* $B(0,R) \setminus \{0\}$*. It is good to consider the function* $h(x) := 2M \frac{\log(\frac{|x|}{R})}{\log(\frac{r}{R})}$*. Why can we say that* $|w(x)| \leq |h(x)|$ *for* $r < |x| < R$*? What happens when* $r \to 0$*?]*

This homework assignment is due in class on Monday, October 17. Good Luck!