MATH 644, FALL 2011, HOMEWORK 1

Exercise 1. *(Duhamel's principle) Consider the differential operator* $L = \sum_\alpha c_\alpha D_x^\alpha$ *on* $\mathbb{R}^n$*. Here,* $c_\alpha \in \mathbb{R}$ *are constants. Suppose that the solution to:*

$$\begin{cases} \partial_t u + L(u) = 0, \text{ on } \mathbb{R}_x^n \times \mathbb{R}_t \\ u|_{t=0} = \Phi, \text{ on } \mathbb{R}_x^n \end{cases} \tag{1}$$

is given by $u(x,t) = S(t)\Phi(x)$.

Show that the solution to:

$$\begin{cases} \partial_t u + L(u) = f(x,t), \text{ on } \mathbb{R}_x^n \times \mathbb{R}_t \\ u|_{t=0} = g(x), \text{ on } \mathbb{R}_x^n \end{cases} \tag{2}$$

is given by:

$$u(x,t) = S(t)g(x) + \int_0^t S(t-\tau)f(x,\tau)d\tau.$$

This is the general version of Duhamel's principle mentioned in class. For the purposes of this exercise, let us assume that Φ, f, g *are smooth and that we can differentiate under the integral sign without additional justification. (HINT: Consider first the case when* $g = 0$*. Be careful when using the Chain rule.)*

Exercise 2. *(Evans, Problem 1 in chapter 2) Write down an explicit formula for a function* u *which solves the initial-value problem:*

$$\begin{cases} \partial_t u + b \cdot \nabla u = 0, \text{ on } \mathbb{R}_x^n \times (0, +\infty)_t \\ u|_{t=0} = g(x), \text{ on } \mathbb{R}_x^n \end{cases} \tag{3}$$

Here, $c \in \mathbb{R}$ *and* $b \in \mathbb{R}^n$ *are constants.*

Exercise 3. *(Evans, Problem 2 in Chapter 2) Prove that Laplace's equation* $\Delta u = 0$ *is rotationally invariant, i.e. if* $\mathcal{O} \in O(n)$ *is an orthogonal* $n \times n$ *matrix, and if* $v = u \circ \mathcal{O}$*, then* $\Delta v = 0$*. We recall that this was a useful insight when we were looking for a fundamental solution of the Laplace operator on* $\mathbb{R}^n$*.*

Exercise 4. *(A vanishing theorem) Suppose that* $u \in L^2(\mathbb{R}^n) \cap C^2(\mathbb{R}^n)$ *solves* $\Delta u = 0$*. Show that* u *is identically equal to zero. (HINT: Use the Mean value property).*

Each problem is worth 5 points. This assignment is due in class on Wednesday, September 21. Good Luck!