PUTNAM SIMULATION EXAM, NOVEMBER 17, 2012.

Exercise 1. *We start from a list of* $2n$ *real numbers* $x_1, \ldots, x_n, y_1, \ldots, y_n$ *and we want to form the scalar product of* $(x_1, \ldots, x_n)$ *and* $(y_1, \ldots, y_n)$*, i.e.*

$$x_1 \cdot y_1 + \cdots x_n \cdot y_n.$$

Each operation of addition and multiplication counts as a single operation. Show that we always need to perform $2n - 1$ *operations in order to calculate the inner product (no matter in what order we perform these operations).*

Exercise 2. *Given* $n \in \mathbb{N}$*, we can write* $(1+\sqrt{2})^n$ *as* $x_n + y_n\sqrt{2}$ *for uniquely determined* $x_n, y_n \in \mathbb{N}$*. For* x_n *and* y_n *defined as above, compute* $x_n^2 - 2y_n^2$ *in terms of* n*.*

Exercise 3. *Given* $a_1, \ldots, a_n, b_1, \ldots, b_n > 0$*, show that:*

$$\sqrt[n]{a_1 \cdots a_n} + \sqrt[n]{b_1 \cdots b_n} \leq \sqrt[n]{(a_1 + b_1) \cdots (a_n + b_n)}.$$

When does equality hold?

Exercise 4. *Let* $\mathcal{A} := \{n \in \mathbb{N};\ n \text{ doesn't contain the digit 9 in its decimal expansion}\}$*. Show that:*

$$\sum_{n \in \mathcal{A}} \frac{1}{n} < +\infty.$$

Exercise 5. *Suppose that* $f : [0,1] \to [0,1]$ *is such that:*

1) *For all* $x, y \in [0,1]$*,* $|f(x) - f(y)| \leq \frac{|f(x)-x|+|f(y)-y|}{2}$*.*
2) *There exist* $a, b \in [0,1]$ *such that* $f(a) = 1, f(b) = 0$*.*

Under these assumptions:

- **i)** *Show that* $f(0) = 1$ *and* $f(1) = 0$*.*
- **ii)** *Suppose that there exists* $x_0 \in [0,1]$ *such that* $f(x_0) = x_0$*. Show that this implies that necessarily* $x_0 = \frac{1}{2}$*.*
- **iii)** *Show that* $f(\frac{1}{2}) = \frac{1}{2}$*, and hence conclude that* f *has a unique fixed point.*

Exercise 6. *Suppose that* $a_n > a_{n-1} > \cdots > a_1 > a_0 > 0$ *are given. Show that the polynomial* $P(z) := a_n z^n + a_{n-1} z^{n-1} + \cdots a_1 z + a_0$ *doesn't have any complex roots which satisfy* $|z| > 1$*.*