SOLUTIONS TO PRACTICE PUTNAM EXAM, OCTOBER 7, 2013.

Each problem is worth 10 points.

Exercise 1. *a) Given a positive integer $k \geq 3$, find coefficients $a, b, c \in \mathbb{R}$, independent of k, such that:*

$$\frac{k^2-2}{k!} = \frac{a}{k!} + \frac{b}{(k-1)!} + \frac{c}{(k-2)!}.$$

b) Show that, for all positive integers $n \geq 3$:

$$3 - \frac{2}{(n-1)!} < \frac{2^2-2}{2!} + \frac{3^2-2}{3!} + \cdots + \frac{n^2-2}{n!} < 3.$$

c) Calculate:

$$\lim_{n \to \infty} \Big(\frac{2^2-2}{2!} + \frac{3^2-2}{3!} + \cdots + \frac{n^2-2}{n!} \Big).$$

Solution:

a) Let k be a positive integer such that $k \geq 3$. We want to find coefficients $a, b, c \in \mathbb{R}$ such that:

$$\frac{k^2-2}{k!} = \frac{a}{k!} + \frac{b}{(k-1)!} + \frac{c}{(k-2)!}.$$

Multiplying through by $k!$, this equation becomes:

$$a + b \cdot k + c \cdot (k^2 - k) = k^2 - 2.$$

In particular:

$$k^2 \cdot c + k \cdot (b - c) + a = k^2 - 2.$$

It follows that this identity will be satisfied for all $k \geq 3$ provided that:

$$c = 1, b - c = 0, a = -2$$

and hence:

$$a = -2, b = 1, c = 1.$$

We note that the a, b, c are independent of k. It follows that we can write:

$$\frac{k^2-2}{k!} = -\frac{2}{k!} + \frac{1}{(k-1)!} + \frac{1}{(k-2)!}.$$

b) Let n be a positive integer such that $n \geq 3$.
From part a), it follows that:

$$I := \frac{2^2-2}{2!} + \frac{3^2-2}{3!} + \cdots + \frac{n^2-2}{n!} =$$

$$= 1 + \Big(-\frac{2}{3!} + \frac{1}{2!} + \frac{1}{1!} \Big) + \Big(-\frac{2}{4!} + \frac{1}{3!} + \frac{1}{2!} \Big) + \Big(-\frac{2}{5!} + \frac{1}{4!} + \frac{1}{3!} \Big) + \cdots +$$

$$+ \cdots + \Big(-\frac{2}{(n-2)!} + \frac{1}{(n-3)!} + \frac{1}{(n-4)!} \Big) + \Big(-\frac{2}{(n-1)!} + \frac{1}{(n-2)!} + \frac{1}{(n-3)!} \Big) + \Big(-\frac{2}{n!} + \frac{1}{(n-1)!} + \frac{1}{(n-2)!} \Big).$$

We note that this sum equals:

$$3 - \frac{1}{(n-1)!} - \frac{2}{n!} = 3 - \frac{n+2}{n!}.$$

It immediately follows that $I < 3$. We now need to check that $I > 3 - \frac{2}{(n-1)!}$. In order to do this, we need to show that $\frac{2}{(n-1)!} > \frac{n+2}{n!}$ whenever $n \geq 3$. The latter inequality is equivalent to $2n > n+2$, which in turn is equivalent to $n > 2$. This inequality is true for $n \geq 3$. It follows that:

$$I > 3 - \frac{2}{(n-1)!},$$

whenever $n \geq 3$.

c) From part b), it follows that:

$$\lim_{n\to\infty} \Big(\frac{2^2-2}{2!} + \frac{3^2-2}{3!} + \cdots + \frac{n^2-2}{n!} \Big) = 3. \ \square$$

Exercise 2. *Let $(x_n)_{n\geq 0}$ be a sequence of non-zero real numbers such that, for all positive natural numbers n, the following identity holds:*

$$x_n^2 - x_{n-1} \cdot x_{n+1} = 1.$$

Show that there exists a real number a such that, for all positive natural numbers n, the following identity holds:

$$x_{n+1} = a \cdot x_n - x_{n-1}.$$

Solution:

Suppose that $n \in \mathbb{N}$. We then observe that:

$$\begin{cases} x_n^2 - x_{n-1} \cdot x_{n+1} = 1 \\ x_{n+1}^2 - x_n \cdot x_{n+2} = 1. \end{cases}$$

If we subtract these two equations, we obtain:

$$x_n^2 + x_n \cdot x_{n+2} = x_{n+1}^2 + x_{n-1} \cdot x_{n+1}.$$

In particular:

$$x_n \cdot (x_n + x_{n+2}) = x_{n+1} \cdot (x_{n-1} + x_{n+1}),$$

which implies:

$$\frac{x_{n+2} + x_n}{x_{n+1}} = \frac{x_{n+1} + x_{n-1}}{x_n}. \tag{1}$$

We can iteratively apply the identity in (1) in order to deduce that for all positive natural numbers n:

$$\frac{x_{n+1} + x_{n-1}}{x_n} = \frac{x_2 + x_0}{x_1}.$$

In particular, if we define $a := \frac{x_2+x_0}{x_1}$. Then a is a real number and it is the case that for all positive natural numbers n:

$$x_{n+1} = a \cdot x_n - x_{n-1}. \ \square$$

Exercise 3. *For non-negative integers n and k, define $Q(n,k)$ to be the coefficient of x^k in the polynomial $(1+x+x^2+x^3)^n$. Show that:*

$$Q(n,k) = \sum_{j=0}^{k} \binom{n}{j} \cdot \binom{n}{k-2j}.$$

Solution: Given a polynomial $p(x)$ and a non-negative integer k, we denote by $[x^k]\,p(x)$ the coefficient of x^k in $p(x)$. With this notation, we observe that:

$$Q(n,k) = [x^k]\,(1+x+x^2+x^3)^n = [x^k]\,\Big((1+x)\cdot(1+x^2)\Big)^n = [x^k]\,\Big((1+x)^n \cdot (1+x^2)^n\Big).$$

We note that this expression equals:

$$\sum_{j=0}^{k} \Big([x^{2j}] \, (1+x^2)^n \Big) \cdot \Big([x^{k-2j}] \, (1+x)^n \Big).$$

By the *Binomial Theorem*, this equals:

$$\sum_{j=0}^{k} \binom{n}{j} \cdot \binom{n}{k-2j},$$

as was claimed. $\square$

Exercise 4. *Suppose that n is a positive integer. Show that:*

$$\Big(\frac{2n-1}{e}\Big)^{\frac{2n-1}{2}} < 1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1) < \Big(\frac{2n+1}{e}\Big)^{\frac{2n+1}{2}}.$$

Solution: We will first prove:

$$\Big(\frac{2n-1}{e}\Big)^{\frac{2n-1}{2}} < 1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1). \tag{2}$$

We take natural logarithms of both sides of (2) and we see that this is equivalent to showing:

$$\frac{2n-1}{2} \cdot \Big(\ln(2n-1) - 1 \Big) < \ln 1 + \ln 3 + \ln 5 + \cdots + \ln(2n-1).$$

In other words, we need to show:

$$(2n-1) \cdot \Big(\ln(2n-1) - 1 \Big) < 2\ln 3 + 2\ln 5 + \cdots + 2\ln(2n-1). \tag{3}$$

We will show (3) by appealing to upper Riemann sums for the natural logarithm function $\ln x$, which is defined for positive x. In particular, we note that:

$$\ln''(x) = -\frac{1}{x^2} < 0.$$

Hence *ln is strictly concave.* Moreover, we know that:

$$\int_a^b \ln(y)\, dy = (a \ln a - a) - (b \ln b - b). \tag{4}$$

We note that the upper Riemann sum corresponding to the integral of ln on the interval $[1, 2n-1]$, with the partition points $1, 3, 5, \ldots, 2n-1$ equals:

$$2\ln 3 + 2\ln 5 + \cdots + 2\ln(2n-1).$$

By the strict concavity of ln, this expression must be

$$> \int_1^{2n+1} \ln(y)\, dy,$$

which by (4) equals:

$$(2n-1)\ln(2n-1) - (2n-1) + 1.$$

In particular:

$$2\ln 3 + 2\ln 5 + \cdots + 2\ln(2n-1) > (2n-1)\ln(2n-1) - (2n-1) = (2n-1) \cdot \Big(\ln(2n-1) - 1 \Big)$$

and the inequality (3) now follows.

By using similar arguments, we will now prove:

$$1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1) < \Big(\frac{2n+1}{e}\Big)^{\frac{2n+1}{1}}. \tag{5}$$

We again take natural logarithms and we deduce that (5) is equivalent to:

$$2\ln 3 + 2\ln 5 + \cdots + 2\ln(2n-1) < (2n+1) \cdot \Big(\ln(2n+1) - 1 \Big). \tag{6}$$

We now observe that the left-hand side equals to the lower Riemann sum corresponding to the integral of ln on the interval $[3, 2n+1]$, with the partition points $3, 5, 7, \cdots, 2n+1$. By strict concavity, this expression is strictly less than $\int_3^{2n+1} \ln(y)\,dy$. Furthermore, we observe that by (4):

$$\int_3^{2n+1} \ln(y)\,dy = (2n+1)\ln(2n+1) - (2n+1) - 3\ln 3 + 3 < (2n+1)\ln(2n+1) - (2n+1) - 3 + 3 =$$

$$= (2n+1)\ln(2n+1) - (2n+1) = (2n+1) \cdot \Big(\ln(2n+1) - 1 \Big).$$

The inequality (6) now follows. $\square$