PRACTICE PUTNAM EXAM, OCTOBER 7, 2013.

No books, notes or calculators. Each problem is worth 10 points.

Exercise 1. *a) Given a positive integer $k \geq 3$, find coefficients $a, b, c \in \mathbb{R}$, independent of k, such that:*

$$\frac{k^2-2}{k!} = \frac{a}{k!} + \frac{b}{(k-1)!} + \frac{c}{(k-2)!}.$$

b) Show that, for all positive integers $n > 2$:

$$3 - \frac{2}{(n-1)!} < \frac{2^2-2}{2!} + \frac{3^2-2}{3!} + \cdots + \frac{n^2-2}{n!} < 3.$$

c) Calculate:

$$\lim_{n\to\infty} \Big(\frac{2^2-2}{2!} + \frac{3^2-2}{3!} + \cdots + \frac{n^2-2}{n!} \Big).$$

Exercise 2. *Let $(x_n)_{n\geq 0}$ be a sequence of non-zero real numbers such that, for all natural numbers n, the following identity holds:*

$$x_n^2 - x_{n-1} \cdot x_{n+1} = 1.$$

Show that there exists a real number a such that, for all natural numbers n, the following identity holds:

$$x_{n+1} = a \cdot x_n - x_{n-1}.$$

Exercise 3. *For non-negative integers n and k, define $Q(n,k)$ to be the coefficient of x^k in the polynomial $(1+x+x^2+x^3)^n$. Show that:*

$$Q(n,k) = \sum_{j=0}^{k} \binom{n}{j} \cdot \binom{n}{k-2j}.$$

Exercise 4. *Suppose that n is a positive integer. Show that:*

$$\Big(\frac{2n-1}{e}\Big)^{\frac{2n-1}{2}} < 1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1) < \Big(\frac{2n+1}{e}\Big)^{\frac{2n+1}{2}}.$$