PRACTICE PUTNAM EXAM, OCTOBER 6, 2012.

No books, notes or calculators. Each problem is worth 10 points.

Exercise 1. *Show that, for all positive integers n:*

$$\frac{1}{n^2}+\big(\frac{1}{n}+\frac{1}{n-1}\big)^2+\cdots+\big(\frac{1}{n}+\frac{1}{n-1}+\cdots+1\big)^2=2n-\big(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\big).$$

Exercise 2. *Consider a function $f:\mathbb{R}\to\mathbb{R}$ such that:*

i) $f(1)=1$.
ii) $f(x+y)=f(x)+f(y)$ *for all* $x,y\in\mathbb{R}$

Under these assumptions:

1) *Show that* $f(0)=0$.
2) *Show that* $f(x)=x$ *for all* $x\in\mathbb{Z}$.
3) *Show that* $f(x)=x$ *for all* $x\in\mathbb{Q}$.
4) *In addition, if f is assumed to be continuous, show that $f(x)=x$ for all $x\in\mathbb{R}$.*
5) *If f is assumed to be monotone instead of continuous, can one make the same conclusion as in 4)?*

Exercise 3. *1) Simplify the expression:*

$$(x^{2^n}+1)(x^{2^{n-1}}+1)\cdots(x^2+1)(x+1).$$

2) Evaluate the limit:

$$\lim_{n\to\infty}(x^{2^n}+1)(x^{2^{n-1}}+1)\cdots(x^2+1)(x+1)$$

when $x=\frac{1}{2}$.

Exercise 4. *Suppose that $ABCD$ is a tetrahedron in 3-space (which is not necessarily regular). At each face $S_j, j=1,\ldots,4$ of the tetrahedron, we draw a vector $\vec{n}_j$ which satisfies:*

i) $\vec{n}_j$ *is perpendicular to* S_j.
ii) $\vec{n}_j$ *points outwards.*
iii) $|\vec{n}_j|$ *equals the surface area of* S_j.

Show that:

$$\vec{n}_1+\vec{n}_2+\vec{n}_3+\vec{n}_4=\vec{0}.$$