PUTNAM PRACTICE PROBLEMS 7

Exercise 1. *We recall that the Fibonacci sequence* (F_n) *is given by:*

$$\begin{cases} F_1 := 1, \ F_2 := 1 \\ F_{n+2} := F_n + F_{n+1} \text{ for } n \geq 1. \end{cases}$$

i) *Show that for all integers* n*, one has:*

$$F_1 + F_2 + \cdots + F_n = F_{n+2} - 1.$$

ii) *Show that every positive integer* m *can be uniquely written in the form:*

$$m = F_{j_1} + F_{j_2} + \cdots + F_{j_k}$$

where the j_ℓ *are positive integers such that*
1) $j_\ell \geq 2$ *for all* $j = 1, 2, \ldots, k$.
2) $j_{\ell+l} > j_\ell + 1$ *for all* $\ell = 1, 2, \ldots, k-1$.

iii) *Give an example of a strictly increasing function* $f : \mathbb{N} \to \mathbb{N}$ *such that:*

$$f(f(n)) = f(n) + n, \text{ for all } n \in \mathbb{N}.$$

Exercise 2. *We are given* $2n$ *distinct points in the plane. Show that one can color* n *points red and* n *points blue (like our school colors) in such a way that:*

i) *Every red point is joined by a line to exactly one red point and every blue point is joined by a line to exactly one blue point.*
ii) *These are the only lines which are drawn.*
iii) *These lines don't intersect.*

[HINT: Think about minimizing the sum of the lengths of the drawn lines.]

Exercise 3. *Let us denote by* $\mathcal{A}$ *the set of all positive integers which are not divisible by the square of any prime number, i.e.* $\mathcal{A} = \{n \in \mathbb{N}; \ q \in \mathbb{N}, q^2 | n \implies q = 1\}$. *Given a positive integer* n*, show that:*

$$\sum_{k \in \mathcal{A}} \left\lfloor \sqrt{\frac{n}{k}} \right\rfloor = n.$$

Here, $\lfloor x \rfloor$ *denotes the largest integer which is less than or equal to* x. *[HINT: Use induction on* n *and think when each term increases as we change from* n *to* $n+1$.*]*