# PUTNAM PRACTICE PROBLEMS 4

**Exercise 1.** *Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that:*

   i) $f(f(n)) = n$.
  ii) $f(f(n+2)+2) = n$.
 iii) $f(0) = 1$.

**Exercise 2.** *Suppose that $f$ and $g$ are non-constant, differentiable and real-valued functions on $\mathbb{R}$ such that:*

$$\begin{cases} f(x + y) = f(x)f(y) - g(x)g(y) \\ g(x + y) = f(x)g(y) + g(x)f(y) \end{cases}$$

*Suppose also that $f'(0) = 0$. Show that, for all $x \in \mathbb{R}$:*

$$(f(x))^2 + (g(x))^2 = 1.$$

**Exercise 3.** *Suppose that $p$ is a prime number. Show that:*

$$\sum_{j=0}^{p} \binom{p}{j} \cdot \binom{p+j}{j} \equiv 2^p + 1 \pmod{p^2}.$$

*[HINT: Recall that $\sum_{j=0}^{p} \binom{p}{j} = 2^p$.]*