PUTNAM PRACTICE PROBLEMS 3

Exercise 1. *(From last week) We recall that the two-dimensional torus $\mathbb{T}^2$ can be obtained by identifying the opposite sides of the square $[0,1] \times [0,1]$. We call a curve $\gamma : \mathbb{R} \to \mathbb{T}^2$ a* ***straight line on the torus*** *if given $t \in \mathbb{R}$, there exists $\epsilon > 0$ such that γ restricted to $[t-\epsilon, t+\epsilon]$ can be identified with a straight line on $[0,1] \times [0,1]$. (In other words, we are just projecting straight lines on $\mathbb{R}^2$ and taking the quotient by translation with $\mathbb{Z}^2$). Can one give a sufficient and necessary condition when a straight line on the torus has a dense image in $\mathbb{T}^2$?*

Exercise 2. *Suppose that p is a prime number. Show that $(p-1)! + 1$ is divisible by p.*

Exercise 3. *Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a twice differentiable function such that:*

$$f''(x) + f(x) = -xg(x)f'(x)$$

for some non-negative function $g : \mathbb{R} \to R$. Show that the function f is bounded.

Exercise 4. *a) Let us denote by $\mathcal{A}$ the set of all positive integers which are not divisible by the square of any prime number, i.e. $\mathcal{A} = \{n \in \mathbb{N};\ q \in \mathbb{N}, q^2 | n \implies q = 1\}$. Given a positive integer n, show that:*

$$\sum_{k \in \mathcal{A}} \left\lfloor \sqrt{\frac{n}{k}} \right\rfloor = n.$$

b) Can one generalize this identity to involve r-th roots, instead of square roots, for a general positive integer $r \geq 2$? How does the definition of the set over which we are summing then change?