PUTNAM PRACTICE PROBLEMS 2

Exercise 1. *Find the minimum value of the expression:*

$$\frac{\left(x+\frac{1}{x}\right)^6-\left(x^6+\frac{1}{x^6}\right)-2}{\left(x+\frac{1}{x}\right)^3+\left(x^3+\frac{1}{x^3}\right)}$$

for $x>0$.

Exercise 2. *Show that all of the numbers* $2^1+1, 2^2+1, 2^{2^2}+1, \dots, 2^{2^n}+1, \dots$ *are pairwise relatively prime.*

For the next exercise, let us first recall the following notation. Given a real number x, let $\lfloor x \rfloor$ equal the largest integer which is less than or equal to x. For instance, $\lfloor 2.5 \rfloor = 2$, and $\lfloor -3.7 \rfloor = -4$. The quantity $\lfloor x \rfloor$ is called the **floor of** x. We also define the **fractional part of** x by $\{x\} := x - \lfloor x \rfloor$. We note that $\{x\} \in [0,1)$.

Exercise 3. *Suppose that* α *is a real number. a) If* α *is rational, show that the set*

$$X_\alpha := \Big\{\{n\alpha\},\, n \in \mathbb{Z}\Big\}$$

is not dense in $[0,1)$.

b) If α *is irrational, show that the set* X_α*, defined as above, is dense in* $[0,1)$.

Exercise 4. *a) We consider a forest in two dimensions such that there is a tree centered at each lattice point of* $\mathbb{Z}^2 = \{(m,n),\, m,n \in \mathbb{Z}\}$ *and such that each tree has radius* $r \in (0, \frac{1}{2})$*. An observer stands at the origin. Does there exist a direction in which it is possible for the observer to see forever? If not, is there a bound as to how far the observer can see?*

b) What happens if the observer is not situated at the origin?

Exercise 5. *Does there exist an integer* n *such that the number* 2^n *in the decimal system starts with the digits* 2012...*?*

Exercise 6. *We recall that the two-dimensional torus* $\mathbb{T}^2$ *can be obtained by identifying the opposite sides of the square* $[0,1]\times[0,1]$*. We call a curve* $\gamma : \mathbb{R} \to \mathbb{T}^2$ *a* ***straight line on the torus*** *if given* $t \in \mathbb{R}$*, there exists* $\epsilon > 0$ *such that* γ *restricted to* $[t-\epsilon, t+\epsilon]$ *can be identified with a straight line on* $[0,1]\times[0,1]$*. (In other words, we are just projecting straight lines on* $\mathbb{R}^2$ *and taking the quotient by translation with* $\mathbb{Z}^2$*). Can one give a sufficient and necessary condition when a straight line on the torus has a dense image in* $\mathbb{T}^2$*?*