PRACTICE HOMEWORK FOR MATH 425, SOLUTIONS

Exercise 1. *Evaluate the integral:*

$$\int_0^{2\pi} e^\theta \sin\theta \, d\theta$$

a) by using Integration by parts.
b) by using complex numbers.

Solution:
a) Method 1: using integration by parts

$$\int_0^{2\pi} e^\theta \sin\theta \, d\theta = \begin{cases} u = e^\theta, & du = e^\theta d\theta \\ dv = \sin\theta \, d\theta, & v = -\cos\theta \end{cases}$$

$$= -e^\theta \cos\theta\Big|_{\theta=0}^{\theta=2\pi} + \int_0^{2\pi} e^\theta \cos\theta \, d\theta = (-e^{2\pi}+1) + \int_0^{2\pi} e^\theta \cos\theta \, d\theta = \begin{cases} u = e^\theta, & du = e^\theta d\theta \\ dv = \cos\theta \, d\theta, & v = \sin\theta \end{cases}$$

$$= (-e^{2\pi}+1) + e^\theta \sin\theta\Big|_{\theta=0}^{\theta=2\pi} - \int_0^{2\pi} e^\theta \sin\theta \, d\theta = (-e^{2\pi}+1) - \int_0^{2\pi} e^\theta \sin\theta \, d\theta$$

Hence,

$$2\int_0^{2\pi} e^\theta \sin\theta \, d\theta = (-e^{2\pi}+1)$$

from where we deduce that the value of the wanted integral is:

$$\frac{-e^{2\pi}+1}{2}$$

b) Method 2: using Complex numbers
We note that:

$$\int_0^{2\pi} e^\theta \sin\theta \, d\theta = Im\Big(\int_0^{2\pi} e^\theta(\cos\theta + i\sin\theta)d\,\theta\Big) = Im\Big(\int_0^{2\pi} e^\theta e^{i\theta} d\,\theta\Big) =$$

$$= Im\Big(\int_0^{2\pi} e^{(1+i)\theta} d\,\theta\Big) = Im\frac{1}{1+i}e^{(1+i)\theta}\Big|_{\theta=0}^{\theta=2\pi} = Im\Big(\frac{1}{1+i}(e^{2\pi}-1)\Big)$$

$$= Im\Big(\frac{1-i}{2}(e^{2\pi}-1)\Big) = \frac{-e^{2\pi}+1}{2} \quad \square.$$

Exercise 2. *Using Euler's formula, rederive the identities:*
a) $\sin(x+y) = \sin x \cos y + \cos x \sin y$.
b) $\cos(x+y) = \cos x \cos y - \sin x \sin y$.

Solution:

We recall that for $x, y \in \mathbb{R}$, one has:

$$e^{i(x+y)} = e^{ix} \cdot e^{iy}.$$

We rewrite both sides by using Euler's formula to obtain:

$$\cos(x+y) + i\sin(x+y) = (\cos x + i\sin x) \cdot (\cos y + i\sin y).$$

It follows that:

$$\cos(x+y) + i\sin(x+y) = (\cos x \cos y - \sin x \sin y) + i(\sin x \cos y + \sin y \cos x).$$

Claims a) and b) now follow by taking real and imaginary parts of both sides $\square$.

Exercise 3. *Find all complex numbers z such that:*
a) $z^6 = 1$.
b) $z^7 = i$.
c) $Re(e^z) > 0$.

Solution:

a) The wanted complex numbers are $z_k = e^{\frac{2\pi i k}{6}} = e^{\frac{k\pi i}{3}} = \cos(\frac{k\pi}{3}) + i\sin(\frac{k\pi}{3})$, for $k = 0, 1, \ldots, 5$.
b) Since $i = e^{\frac{i\pi}{2}}$, we can deduce that the the solutions are given by $z_k = e^{\frac{i\pi}{14} + \frac{2\pi k i}{7}}$, for $k = 0, 1, \ldots, 6$.
c) We write $z = re^{i\theta}$, where $r > 0$ and $\theta \in [0, 2\pi)$. In this way, r and θ are uniquely determined from z. Since $z = r\cos\theta + i\sin\theta$, we deduce that:

$$e^z = e^{r(\cos\theta + i\sin\theta)} = e^{r\cos\theta} \cdot e^{ir\sin\theta} = e^{r\cos\theta} \cdot \big(\cos(r\sin\theta) + i\sin(r\sin\theta)\big)$$

Since $e^{r\cos\theta}$ is a positive real number, the condition we need to satisfy is $\cos(r\sin\theta) > 0$. An equivalent way to write this is to say that there exists $k \in \mathbb{Z}$ such that:

$$r\sin\theta \in (-\frac{\pi}{2} + 2k\pi, \frac{\pi}{2} + 2k\pi). \ \square$$

Exercise 4. *a) For what $c \in \mathbb{R}$ does there exist a non-zero function $w : [0, 2\pi] \to \mathbb{C}$ such that:*

$$w'' - c^2 w = 0$$

and such that $w(0) = w(2\pi) = 0$?
b) What if w instead solves $w'' + c^2 w = 0$ (again with the assumption that $w(0) = w(2\pi) = 0$)?

Solution:

Let us first suppose that $c \neq 0$. From ODE theory, we know that $w = a_1 e^{ct} + a_2 e^{-ct}$ for some (complex numbers) a_1, a_2. The condition $w(0) = w(2\pi) = 0$ then implies that:

$$\begin{cases} a_1 + a_2 = 0 \\ a_1 e^{2\pi c} + a_2 e^{-2\pi c} = 0 \end{cases}$$

From the above two equations, it follows that $a_1 = a_2 = 0$ and so w is identically zero. If $c = 0$, then $w = a_1 + a_2 t$. In this case, $w(0) = 0$ implies that $a_1 = 0$ and $w(2\pi) = 0$ implies that $a_2 = 0$, and so w is again identically zero. Hence, in a), it is not possible to find such a function w.

b) We now consider what happens when $w'' + c^2 w = 0$. Based on part *a*), we need to assume that $c \neq 0$. In this case, we recall that $w(t) = a_1\cos(ct) + a_2\sin(ct)$. Since $w(0) = a_1 = 0$, it follows that $w(t) = a_2\sin(ct)$. We then obtain that $w(2\pi) = a_2\sin(2\pi c)$. Since we want $a_2 \neq 0$ (since otherwise, w is identically zero), it follows that we need to have $\sin(2\pi c) = 0$, and hence $2\pi c = k\pi$ for some $k \in \mathbb{Z}$. Consequently, $c = \frac{k}{2}$ for some $k \in \mathbb{Z} \setminus \{0\}$. $\square$

Exercise 5. *Suppose that $w : [0, +\infty) \to \mathbb{R}$ solves the ODE:*

$$aw'' + bw' + cw = 0 \tag{1}$$

for some constants a, b, c. Furthermore, we assume that $b \geq 0$.
a) Let us define the ***Energy*** *to be:*

$$E(t) := \frac{1}{2}\big[a(w'(t))^2 + c(w(t))^2\big].$$

Without solving the ODE (1), show that $E'(t) \leq 0$.
b) Under the additional assumption that $a > 0$ and $c > 0$, show that $w(0) = 0$ and $w'(0) = 0$ implies that $w(t) = 0$ for all $t \geq 0$.
c) Assume again that $a > 0$ and $c > 0$. Show that if w_1 and w_2 solve the ODE (1) and if $w_1(0) =$

$w_2(0), w_1'(0) = w_2'(0)$, *then one can deduce that* $w_1(t) = w_2(t)$ *for all* $t \geq 0$. *In this way, we obtain uniqueness of solutions to* (1).

Solution:

a) We use the product rule to calculate: $E'(t) = aw'w'' + cww'$. We can now use the ODE to deduce that $w'' = -bw' - cw$. Hence:

$$E'(t) = w'(-bw' - cw) + cww' = -b(w')^2 \leq 0$$

since $b \geq 0$. In other words, $E(t)$ is a decreasing function of t on $[0, +\infty)$.

b) By assumption $E(0) = \frac{1}{2}\big[a(w'(0))^2 + c(w(0))^2\big] = 0$. Since $a, c > 0$, it follows that $E(t)$ is non-negative. Finally, from part a), it follows that $E(t)$ is a decreasing function on $[0, +\infty)$, hence $E(t)$ is identically zero on $[0, +\infty)$. In particular, since both a and c are positive, it follows that $w(t) = 0$ for all $t \geq 0$.

c) If w_1 and w_2 solve the ODE, then so does $w := w_1 - w_2$. The function w then satisfies the conditions of part b) and the claim follows. $\square$