MATH 425, PRACTICE FINAL EXAM SOLUTIONS.

Exercise 1. *a) Is the operator $\mathcal{L}_1$ defined on smooth functions of (x,y) by $\mathcal{L}_1(u) := u_{xx} + \cos(u)$ linear?*

b) Does the answer change if we replace the operator $\mathcal{L}_1$ by the operator $\mathcal{L}_2$, which is is given by:

$$\mathcal{L}_2(u) = \cos(x^2 y) \cdot u_{xx} + e^{xy^2} \cdot u \ ?$$

c) Find the general solution of the PDE $(1+x^2)u_x + u_y = 0$ by using the method of characteristics.

d) Give a rigorous proof that the solution you found in part c) is the general solution to the given PDE.

Solution:

a) We note that $\mathcal{L}_1(0) = 1 \neq 0$, which implies that the operator is not linear. Namely, for a linear operator T, we know that $T(0) = 0$ if we substitute $a = b = 0$ into the definition of linearity.

b) Given smooth functions u, v and constants a, b, we compute:

$$\mathcal{L}_2(au+bv) = \cos(x^2y)\cdot\big(au+bv\big)_{xx}+e^{xy^2}\big(au+bv\big) = a(\cos(x^2y)u_{xx}+e^{xy^2}u)+b(\cos(x^2y)v_{xx}+e^{xy^2}v) =$$

$$= a\mathcal{L}_1(u) + b\mathcal{L}_2(v).$$

Hence, $\mathcal{L}_2$ is linear.

c) We use the method of characteristics. The equation can be rewritten as:

$$u_x + \frac{1}{1+x^2}u_y = 0.$$

The characteristic equation can then be written as:

$$\frac{dy}{dx} = \frac{1}{1+x^2}.$$

This ODE has the solution:

$$y(x) = \arctan(x) + C$$

for some constant $C \in \mathbb{R}$. Consequently, the general solution to the PDE is given by:

$$u(x,t) = f(y - \arctan(x))$$

for some differentiable function $f : \mathbb{R} \to \mathbb{R}$

d) Let us fix a constant $C \in \mathbb{R}$. We look at the parametrixed curve $z : \mathbb{R} \to \mathbb{R}^2$ which is defined by:

$$z(t) = (t, \arctan(t) + C).$$

We need to show that u is constant along this parametrized curve. In order to do this, we use the Chain Rule in order to compute:

$$\frac{d}{dt}u(z(t)) = \frac{d}{dt}u(t,\arctan(t)+C) = \big(u_x\big)(t,\arctan(t)+C) + \frac{1}{1+t^2}\big(u_y\big)(t,\arctan(t)+C) =$$

$$= \big(u_x + \frac{1}{1+x^2}u_y\big)(t,\arctan(t)+C) = 0.$$

The claim now follows. $\square$

Exercise 2. *(Uniqueness for the heat equation with Neumann boundary conditions)*
Use the energy method to show uniqueness of solutions for the following boundary value problem:

$$\begin{cases} u_t - ku_{xx} = f(x,t) \text{ for } 0 < x < \ell, t > 0 \\ u(x,0) = \phi(x) \text{ for } 0 < x < \ell \\ u_x(0,t) = g(t),\, u_x(\ell,t) = h(t) \text{ for } t > 0. \end{cases}$$

Here, $\ell > 0$ *and the functions* f, g, h, ϕ *are assumed to be smooth.*

Solution:

It suffices to consider the following boundary value problem:

$$\begin{cases} w_t - kw_{xx} = 0 \text{ for } 0 < x < \ell, t > 0 \\ w(x,0) = 0 \text{ for } 0 < x < \ell \\ w_x(0,t) = 0,\, w_x(\ell,t) = 0 \text{ for } t > 0. \end{cases}$$

and to show that necessarily $w = 0$.

We define the energy to be:

$$E(t) := \int_0^\ell \Big(w(x,t)\Big)^2 \, dx.$$

By differentiating under the integral sign, it follows that:

$$\frac{d}{dt}E(t) = 2\int_0^\ell w(x,t) \cdot w_t(x,t)\, dx.$$

By using the equation, this expression equals:

$$2\int_0^\ell w(x,t) \cdot kw_{xx}(x,t)\, dx.$$

We now integrate by parts in x in order to deduce that this equals:

$$2kw(x,t) \cdot w_x(x,t)\Big|_{x=0}^{x=\ell} - 2\int_0^\ell \Big(w_x(x,t)\Big)^2 dx = -2\int_0^\ell \Big(w_x(x,t)\Big)^2 dx \leq 0.$$

Hence, $E(t)$ is decreasing in t. In particular, since $E(0) = 0$ and since $E(t) \geq 0$, it follows that $E(t) = 0$ for all $t \geq 0$. Hence, $w = 0$, as was claimed. $\square$.

Exercise 3. *(A maximum principle bound for solutions of Poisson's equation)*
Let $\Omega \subseteq \mathbb{R}^3$ *be a bounded domain and suppose that* $f : \overline{\Omega} \to \mathbb{R}$ *is continuous on* $\overline{\Omega}$ *and that* $g : \partial\Omega \to \mathbb{R}$ *is continuous on* $\partial\Omega$. *Suppose that the function* u *solves the Poisson equation on* Ω*:*

$$\begin{cases} \Delta u = f \text{ on } \Omega \\ u = g \text{ on } \partial\Omega. \end{cases}$$

a) Assuming that the function u *exists, is smooth on* Ω*, and continuous on* $\overline{\Omega}$*, show that it is uniquely determined.*

b) Show that for sufficiently large $\lambda > 0$*, depending on the function* f*, the function* $u + \lambda x^2$ *is subharmonic, i.e.* $\Delta(u + \lambda x^2) \geq 0$.

c) Use the maximum principle for subharmonic functions (from Homework 6) in order to deduce that, there exists a constant $C > 0$*, depending only on* Ω *such that, for all* $x \in \Omega$*, the following bound holds:*

$$u(x) \leq \max_{\partial\Omega} g + C \max_{\overline{\Omega}} |f|.$$

Solution:

a) We need to prove uniqueness of solutions for the boundary value problem. It suffices to show that, if w solves:

$$\begin{cases} \Delta w = 0 \text{ on } \Omega \\ w = 0 \text{ on } \partial\Omega. \end{cases}$$

then $w = 0$ on $\overline{\Omega}$.
This claim follows immediately from the Maximum principle for solutions to the Laplace equation. In Homework 6, Exercise 1, we saw how to deduce this claim by using the Energy method.

b) We recall that, on $\mathbb{R}^3$, it is true that:

$$\Delta(x^2) = \Delta(x_1^2 + x_2^2 + x_3^2) = 6.$$

Hence:

$$\Delta(u + \lambda x^2) = \Delta u + \lambda \cdot \Delta(x^2) = f + 6\lambda.$$

This quantity is non-negative if

$$\lambda \geq \frac{\max_{\overline{\Omega}} |f|}{6}.$$

For such a λ, the function $u + \lambda x^2$ is subharmonic.

c) Let us take λ as in part b). We note that the function $u + \lambda x^2$ solves the following boundary value problem:

$$\begin{cases} \Delta u + \lambda x^2 = f + \lambda x^2 \text{ on } \Omega \\ u + \lambda x^2 = g + \lambda x^2 \text{ on } \partial\Omega. \end{cases}$$

By using the Maximum principle for subharmonic functions (Exercise 3 from Homework 6), it follows that for all $x \in \Omega$, the following bound holds:

$$u(x) + \lambda x^2 \leq \max_{y \in \partial\Omega} \Big(g(y) + \lambda y^2 \Big) = g(y_0) + \lambda y_0^2$$

for some $y_0 \in \partial\Omega$. In particular, since $\lambda \geq 0$:

$$u(x) \leq g(y_0) + \lambda(y_0^2 - x^2) \leq g(y_0) + \lambda y_0^2.$$

Since Ω is bounded, we can find a constant $K > 0$ depending only on Ω such that for all $y \in \partial\Omega$, one has: $y^2 \leq K$. Hence, it follows that for all $x \in \Omega$:

$$u(x) \leq \max_{\partial\Omega} g + \lambda \cdot K.$$

We now note that, if $f \geq 0$, we can take $\lambda = 0$ and the claim will follow from the maximum principle for subharmonic functions.
If it is not the case that $f \geq 0$, we can take $\lambda = \frac{\max_{\overline{\Omega}} |f|}{6} > 0$, and $C = \frac{K}{6}$ to obtain that for all $x \in \Omega$:

$$u(x) \leq \max_{\partial\Omega} g + C \max_{\overline{\Omega}} |f|.$$

We note that the constant $C > 0$ depends only on the domain Ω. $\square$

Exercise 4. *(The sign of the Green's function)*
Suppose that $\Omega \subseteq \mathbb{R}^3$ is a bounded domain. Given $x_0 \in \Omega$, we consider the Green's function $G(x, x_0)$. We recall from the definitions that then the function:

$$H(x, x_0) := G(x, x_0) + \frac{1}{4\pi|x - x_0|}$$

is well-defined and harmonic on all of Ω.

a) Show that, for all $x \in \Omega \setminus \{x_0\}$*:*

$$G(x, x_0) < 0.$$

b) Does the claim still hold if we assume that Ω *is a domain in two-dimensions?*

Solution:

a) We can assume without loss of generality that Ω is connected. If this is not the case, we just consider each connected component of Ω separately.

Let us fix $\epsilon > 0$ small. We define:

$$\Omega_\epsilon := \Omega \setminus B(x_0, \epsilon).$$

If ϵ is sufficiently small, then Ω_ϵ is also connected.
Let us note that the function $G(x, x_0) = H(x, x_0) - \frac{1}{4\pi|x - x_0|}$ is harmonic on Ω_ϵ. We recall that $H(x, x_0)$ is bounded for $x \in \Omega$ and that $\lim_{x \to x_0} \frac{1}{|x - x_0|} = +\infty$. In particular, it follows that if we choose $\epsilon > 0$ sufficiently small, we obtain that:

$$G(x, x_0) < 0 \text{ for } x \in \partial B(x_0, \epsilon).$$

By construction of the Green's function, we know that:

$$G(x, x_0) = 0 \text{ for } x \in \partial\Omega.$$

We no use the (weak) Maximum principle for the harmonic function $G(x, x_0)$ on Ω_ϵ in order to deduce that:

$$G(x, x_0) \leq 0 \text{ for } x \in \Omega_\epsilon.$$

We can say even more if we use the Strong Maximum principle. Namely, since $G(x, x_0)$ is not identically equal to zero on Ω_ϵ, it follows that:

$$G(x, x_0) < 0 \text{ for } x \in \Omega_\epsilon.$$

We can now let $\epsilon \to 0$ in order to deduce that:

$$G(x, x_0) < 0 \text{ for } x \in \Omega \setminus \{x_0\}.$$

b) In two dimensions, we recall from Exercise 1 on Homework 8 that, in the definition of the Green's function, the function $H(x, x_0) := G(x, x_0) - \frac{1}{2\pi} \log|x - x_0|$ has to be harmonic. The same argument as above applies since we know that $\lim_{x \to x_0} \log|x - x_0| = -\infty$. In particular, we obtain that $G(x, x_0) < 0$ for $x \in \Omega \setminus \{x_0\}$. $\Box$

Exercise 5. *(Finite speed of propagation for the wave equation without using the d'Alembert formula)*

Suppose that $u : \mathbb{R}_x \times \mathbb{R}_t$ *solves the initial value problem for the wave equation:*

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 \textit{ on } \mathbb{R}_x \times \mathbb{R}_t \\ u|_{t=0} = \phi,\, u_t|_{t=0} = \psi \textit{ on } \mathbb{R}_x. \end{cases}$$

Given $x_0 \in \mathbb{R}$ *and* $t_0 > 0$*, we define the cone:*

$$K(x_0, t_0) := \{(x, t) \in \mathbb{R} \times [0, t_0],\ |x - x_0| \leq c(t_0 - t)\}.$$

Furthermore, for $0 \leq t \leq t_0$*, we define the local energy:*

$$e(t) := \frac{1}{2} \int_{x_0 - c(t_0 - t)}^{x_0 + c(t_0 - t)} \Big[(u_t(x, t))^2 + c^2 (u_x(x, t))^2 \Big]\, dx.$$

a) Show that:

$$e'(t) = \int_{x_0-c(t_0-t)}^{x_0+c(t_0-t)} \Big(u_t(x,t)u_{tt}(x,t) + c^2 u_x(x,t)u_{xt}(x,t)\Big)\,dx$$
$$-\frac{1}{2}c\Big(u_t(x_0+c(t_0-t),t)\Big)^2 - \frac{1}{2}c^3\Big(u_x(x_0+c(t_0-t),t)\Big)^2$$
$$-\frac{1}{2}c\Big(u_t(x_0-c(t_0-t),t)\Big)^2 - \frac{1}{2}c^3\Big(u_x(x_0-c(t_0-t),t)\Big)^2.$$

b) Use the equation and integrate by parts to show that:

$$e'(t) \leq 0$$

for all $0 \leq t \leq t_0$.

c) Suppose that $\phi = \psi = 0$. *Show that then:*

$$e(t) = 0$$

for all $0 \leq t \leq t_0$.

d) Deduce that, when $\phi = \psi = 0$, *the function* u *equals zero on* $K(x_0, t_0)$. *In this way, we can deduce the domain of dependence result from class without having to use the explicit formula for* u.

Remark: *This method of proof, which doesn't rely on the d'Alembert formula, can be generalized to show that the principle of finite speed of propagation for the wave equation holds in higher dimensions.*

Solution:

a) The formula

$$e'(t) = \int_{x_0-c(t_0-t)}^{x_0+c(t_0-t)} \Big(u_t(x,t)u_{tt}(x,t) + c^2 u_x(x,t)u_{xt}(x,t)\Big)\,dx$$
$$-\frac{1}{2}c\Big(u_t(x_0+c(t_0-t),t)\Big)^2 - \frac{1}{2}c^3\Big(u_x(x_0+c(t_0-t),t)\Big)^2$$
$$-\frac{1}{2}c\Big(u_t(x_0-c(t_0-t),t)\Big)^2 - \frac{1}{2}c^3\Big(u_x(x_0-c(t_0-t),t)\Big)^2$$

is obtained by first differentiating under the integral sign in order to obtain the first integral. The remaining four terms are obtained from differentiating the occurrence of t in the boundary of the integral and by using the Fundamental theorem of Calculus. Here, we have an additional factor of $-c$ due to the Chain Rule.

b) We use the equation in order to deduce that:

$$e'(t) = \int_{x_0-c(t_0-t)}^{x_0+c(t_0-t)} \Big(c^2 u_t(x,t)u_{xx}(x,t) + c^2 u_x(x,t)u_{xt}(x,t)\Big)\,dx$$
$$-\frac{1}{2}c\Big(u_t(x_0+c(t_0-t),t)\Big)^2 - \frac{1}{2}c^3\Big(u_x(x_0+c(t_0-t),t)\Big)^2$$
$$-\frac{1}{2}c\Big(u_t(x_0-c(t_0-t),t)\Big)^2 - \frac{1}{2}c^3\Big(u_x(x_0-c(t_0-t),t)\Big)^2.$$

We can now integrate by parts in x in the first term in the integral to deduce that:

$$e'(t) = \int_{x_0-c(t_0-t)}^{x_0+c(t_0-t)} \Big(-c^2 u_{tx}(x,t)u_x(x,t) + c^2 u_x(x,t)u_{xt}(x,t)\Big)\,dx + c^2 u_t(x,t)\cdot u_x(x,t)\Big|_{x=x_0-c(t_0-t)}^{x=x_0+c(t_0-t)}$$

$$-\frac{1}{2}c\Big(u_t(x_0+c(t_0-t),t)\Big)^2-\frac{1}{2}c^3\Big(u_x(x_0+c(t_0-t),t)\Big)^2$$
$$-\frac{1}{2}c\Big(u_t(x_0-c(t_0-t),t)\Big)^2-\frac{1}{2}c^3\Big(u_x(x_0-c(t_0-t),t)\Big)^2.$$

Hence:

$$e'(t)=\Big[c^2u_t(x_0+c(t_0-t),t)\cdot u_x(x_0+c(t_0-t),t)-\frac{1}{2}c\Big(u_t(x_0+c(t_0-t),t)\Big)^2-\frac{1}{2}c^3\Big(u_x(x_0+c(t_0-t),t)\Big)^2\Big]$$
$$+\Big[-c^2u_t(x_0-c(t_0-t),t)\cdot u_x(x_0-c(t_0-t),t)-\frac{1}{2}c\Big(u_t(x_0-c(t_0-t),t)\Big)^2-\frac{1}{2}c^3\Big(u_x(x_0-c(t_0-t),t)\Big)^2\Big].$$

We now recall the fact that for all $A,B\in\mathbb{R}$, the following inequality holds:

$$AB\leq\frac{A^2+B^2}{2}.$$

This inequality follows from the fact that:

$$\frac{1}{2}(A-B)^2\geq 0.$$

Consequently:

$$c^2u_t(x_0+c(t_0-t),t)\cdot u_x(x_0+c(t_0-t),t)-\frac{1}{2}c\Big(u_t(x_0+c(t_0-t),t)\Big)^2-\frac{1}{2}c^3\Big(u_x(x_0+c(t_0-t),t)\Big)^2\leq 0$$

if we take $A=\sqrt{c}\cdot u_t(x_0+c(t_0-t),t)$ and $B=(\sqrt{c})^3\cdot u_x(x_0+c(t_0-t),t)$.
Moreover, we can deduce that:

$$-c^2u_t(x_0-c(t_0-t),t)\cdot u_x(x_0-c(t_0-t),t)-\frac{1}{2}c\Big(u_t(x_0-c(t_0-t),t)\Big)^2-\frac{1}{2}c^3\Big(u_x(x_0-c(t_0-t),t)\Big)^2\leq 0$$

if we take $A=-\sqrt{c}\cdot u_t(x_0-c(t_0-t),t)$ and $B=(\sqrt{c})^3\cdot u_x(x_0-c(t_0-t),t)$.
It follows that:

$$e'(t)\leq 0 \text{ for } 0\leq t\leq t_0.$$

c) If $\psi=\phi=0$, it follows that $e(0)=0$. Since $e(t)$ is non-negative and $e'(t)$ is non-increasing, it follows that $e(t)=0$ for all $0\leq t\leq t_0$.

d) From part c, it follows that $u_t=0_x=0$ on $K(x_0,t_0)$. Hence, u is constant on $K(x_0,t_0)$. Since $\phi=0$, it follows that u equals to zero on $K(x_0,t_0)$. In particular, we obtain the domain of dependence theorem from class without having to use the explicit formula for $u(x,t)$. $\Box$