MATH 425, PRACTICE FINAL EXAM.

There are five problems on the practice exam, as there will be on the real final exam. The fifth question on the final exam will be a fact about the wave equation from the class or from the homework.

Exercise 1. *a) Is the operator $\mathcal{L}_1$ defined on smooth functions of (x,y) by $\mathcal{L}_1(u) := u_{xx} + \cos(u)$ linear?*

b) Does the answer change if we replace the operator $\mathcal{L}_1$ by the operator $\mathcal{L}_2$, which is is given by:

$$\mathcal{L}_2(u) = \cos(x^2 y) \cdot u_{xx} + e^{xy^2} \cdot u \text{ ?}$$

c) Find the general solution of the PDE $(1+x^2)u_x + u_y = 0$ by using the method of characteristics.

d) Give a rigorous proof that the solution you found in part c) is the general solution to the given PDE.

Exercise 2. *(Uniqueness for the heat equation with Neumann boundary conditions)*
Use the energy method to show uniqueness of solutions for the following boundary value problem:

$$\begin{cases} u_t - k u_{xx} = f(x,t) \text{ for } 0 < x < \ell, t > 0 \\ u(x,0) = \phi(x) \text{ for } 0 < x < \ell \\ u_x(0,t) = g(t),\, u_x(\ell,t) = h(t) \text{ for } t > 0. \end{cases}$$

Here, $\ell > 0$ and the functions f, g, h, ϕ are assumed to be smooth. [HINT: Consider the energy $E[w] := \int_0^\ell w^2\,dx$.]

Exercise 3. *(A maximum principle bound for solutions of Poisson's equation)*
Let $\Omega \subseteq \mathbb{R}^3$ be a bounded domain and suppose that $f : \overline{\Omega} \to \mathbb{R}$ is continuous on $\overline{\Omega}$ and that $g : \partial\Omega \to \mathbb{R}$ is continuous on $\partial\Omega$. Suppose that the function u solves the Poisson equation on Ω:

$$\begin{cases} \Delta u = f \text{ on } \Omega \\ u = g \text{ on } \partial\Omega. \end{cases}$$

a) Assuming that the function u exists, is smooth on Ω, and continuous on $\overline{\Omega}$, show that it is uniquely determined.

b) Show that for sufficiently large $\lambda > 0$, depending on the function f, the function $u + \lambda x^2$ is subharmonic, i.e. $\Delta(u + \lambda x^2) \geq 0$.

c) Use the maximum principle for subharmonic functions (from Homework 6) in order to deduce that, there exists a constant $C > 0$, depending only on Ω such that, for all $x \in \Omega$, the following bound holds:

$$u(x) \leq \max_{\partial\Omega} g + C \max_{\overline{\Omega}} |f|.$$

Exercise 4. *(The sign of the Green's function)*
Suppose that $\Omega \subseteq \mathbb{R}^3$ is a bounded domain. Given $x_0 \in \Omega$, we consider the Green's function $G(x, x_0)$. We recall from the definitions that then the function:

$$H(x,x_0) := G(x,x_0) + \frac{1}{4\pi|x - x_0|}$$

is well-defined and harmonic on all of Ω.

a) Show that, for all $x \in \Omega \setminus \{x_0\}$*:*

$$G(x, x_0) < 0.$$

[HINT: Isolate the singularity at x_0 *and apply the maximum principle.]*

b) Does the claim still hold if we assume that Ω *is a domain in two-dimensions?*

Exercise 5. *(Finite speed of propagation for the wave equation without using the d'Alembert formula)*

Suppose that $u : \mathbb{R}_x \times \mathbb{R}_t$ *solves the initial value problem for the wave equation:*

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 \textit{ on } \mathbb{R}_x \times \mathbb{R}_t \\ u|_{t=0} = \phi,\ u_t|_{t=0} = \psi \textit{ on } \mathbb{R}_x. \end{cases}$$

Given $x_0 \in \mathbb{R}$ *and* $t_0 > 0$*, we define the cone:*

$$K(x_0, t_0) := \{(x, t) \in \mathbb{R} \times [0, t_0],\ |x - x_0| \leq c(t_0 - t)\}.$$

Furthermore, for $0 \leq t \leq t_0$*, we define the local energy:*

$$e(t) := \frac{1}{2} \int_{x_0 - c(t_0 - t)}^{x_0 + c(t_0 - t)} \Big[(u_t(x,t))^2 + c^2 (u_x(x,t))^2 \Big]\, dx.$$

a) Show that:

$$\begin{aligned} e'(t) = & \int_{x_0 - c(t_0 - t)}^{x_0 + c(t_0 - t)} \Big(u_t(x,t) u_{tt}(x,t) + c^2 u_x(x,t) u_{xt}(x,t) \Big)\, dx \\ & - \frac{1}{2} c \Big(u_t(x_0 + c(t_0 - t), t) \Big)^2 - \frac{1}{2} c^3 \Big(u_x(x_0 + c(t_0 - t), t) \Big)^2 \\ & - \frac{1}{2} c \Big(u_t(x_0 - c(t_0 - t), t) \Big)^2 - \frac{1}{2} c^3 \Big(u_x(x_0 - c(t_0 - t), t) \Big)^2. \end{aligned}$$

b) Use the equation and integrate by parts to show that:

$$e'(t) \leq 0$$

for all $0 \leq t \leq t_0$*. [HINT: In the proof, one should recall that* $AB \leq \frac{A^2 + B^2}{2}$ *for all* $A, B \in \mathbb{R}$*.]*

c) Suppose that $\phi = \psi = 0$*. Show that then:*

$$e(t) = 0$$

for all $0 \leq t \leq t_0$*.*

d) Deduce that, when $\phi = \psi = 0$*, the function* u *equals zero on* $K(x_0, t_0)$*. In this way, we can deduce the domain of dependence result from class without having to use the explicit formula for* u*.*

Remark: *This method of proof, which doesn't rely on the d'Alembert formula, can be generalized to show that the principle of finite speed of propagation for the wave equation holds in higher dimensions.*