MATH 425, HOMEWORK 8

This homework is due in class on Tuesday, April 23. Each problem is worth 10 points.

Exercise 1. *(Green's functions in two dimensions)*

Let $\Omega \subseteq \mathbb{R}^2$ be a bounded domain. Suppose that $u : \Omega \to \mathbb{R}$ is a harmonic function which extends continuously to $\overline{\Omega} = \Omega \cup \partial\Omega$.

a) Prove that, for all $x_0 \in \Omega$:

$$u(x_0) = \frac{1}{2\pi} \int_{\partial\Omega} \Big[u(x) \cdot \frac{\partial}{\partial n} \log|x - x_0| - \frac{\partial u}{\partial n}(x) \cdot \log|x - x_0| \Big] \, ds.$$

Here, ds denotes the arclength element on $\partial\Omega$ (recall that each connected component of $\partial\Omega$ is a smooth curve).

b) Formulate a definition for the Green's function for the Laplace equation on the two-dimensional domain Ω as in part a).

c) Show that, for fixed $x_0 \in \Omega$, and for the right definition of the Green's function $G(x, x_0)$, it is true that:

$$u(x_0) = \int_{\partial\Omega} u(x) \cdot \frac{\partial G(x, x_0)}{\partial n} \, dS$$

for all harmonic functions u as in part a).

Exercise 2. *(An averaging property for smooth functions)*
Suppose that $\phi : \mathbb{R}^3 \to \mathbb{R}$ is a smooth function which equals zero outside of some ball centered at the origin.

a) Prove that:

$$\phi(0) = -\frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{1}{|x|} \cdot \Delta\phi(x) \, dx.$$

[HINT: Use Green's second identity. Be careful to isolate the singularity.]

b) Why is identity in part a) immediate if the function ϕ is assumed to be harmonic?

Exercise 3. *(Uniqueness of Green's functions)*

Suppose that $\Omega \subseteq \mathbb{R}^3$ is a bounded domain. Suppose that, for given $x_0 \in \Omega$, the functions $G^1(x, x_0)$ and $G^2(x, x_0)$, defined for $x \in \Omega \setminus \{x_0\}$, satisfy the conditions of the Green's function stated in class.

Prove that:

$$G^1(x, x_0) = G^2(x, x_0)$$

for all $x \in \Omega \setminus \{x_0\}$. In other words, the Green's function is uniquely defined.

Exercise 4. *(Equipartition of energy for the wave equation) Suppose that $g, h : \mathbb{R} \to \mathbb{R}$ are smooth functions which vanish outside of some interval of finite length and let $u \in C^2(\mathbb{R} \times [0, +\infty))$ solve the initial value problem for the wave equation in one dimension:*

$$(1) \qquad \begin{cases} u_{tt} - u_{xx} = 0 \text{ on } \mathbb{R} \times (0, +\infty) \\ u = g, u_t = h \text{ on } \mathbb{R} \times \{t = 0\}. \end{cases}$$

Note that, in this case the constant c is assumed to equal 1.

*The **kinetic energy** of the solution u is defined by:*

$$k(t) := \frac{1}{2} \int_{-\infty}^{+\infty} u_t^2(x,t) dx$$

*and the **potential energy** of u is defined by:*

$$p(t) := \frac{1}{2} \int_{-\infty}^{+\infty} u_x^2(x,t) dx.$$

a) Show that $k(t) + p(t)$ is constant in time by using the formula from class:

$$u(x,t) = f(x-t) + g(x+t).$$

Hence, the total energy is conserved in time. We recall that, in class, we proved this fact directly by using the equation.

b) Moreover, show that $k(t) = p(t)$ for sufficiently large t. In other words, the total energy gets equally partitioned into the kinetic and potential part over a sufficiently long time.