MATH 425, HOMEWORK 7, SOLUTIONS

Each problem is worth 10 points.

Exercise 1. *(An alternative derivation of the mean value property in 3D)*
Suppose that u is a harmonic function on a domain $\Omega \subseteq \mathbb{R}^3$ and suppose that $B(x,R) \subseteq \Omega$. We want to show that:

$$u(x) = \frac{1}{4\pi r^2} \int_{\partial B(x,r)} u(y) dS(y)$$

for all $r \in (0,R)$. This is the mean value property for harmonic functions in three dimensions. In class, we showed the analogous claim in two dimensions by using Poisson's formula. In this exercise, we outline how to give an alternative proof of the mean value property.

a) Define the function $g : (0,R) \to \mathbb{R}$ by:

$$g(r) := \frac{1}{4\pi r^2} \int_{\partial B(x,r)} u(y)\, dS(y).$$

By using a change of variables which takes $B(x,R)$ to $B(0,1)$, show that:

$$g(r) = \frac{1}{4\pi} \int_{\partial B(0,1)} u(x+rz)\, dS(z).$$

(In this way the domain of integration no longer depends on r.)

b) Show that:

$$g'(r) = \frac{1}{4\pi r^2} \int_{\partial B(x,r)} \nabla u(y) \cdot \vec{n}(y)\, dS(y)$$

where $\vec{n}(y)$ is the outward-pointing unit normal vector to $\partial B(x,r)$ at the point $y \in \partial B(x,r)$.
[HINT: Differentiate under the integral sign and then undo the change of variables in a).]

d) Use the Divergence Theorem and the Laplace equation to deduce that $g'(r) = 0$.

e) What is $\lim_{r \to 0} g(r)$? (Recall that u is smooth. In particular, it is continuous).

f) Conclude the proof of the mean value property.

g) [Extra Credit (2 points)] Show that if u is assumed to be continuous and subharmonic on Ω (i.e. $\Delta u \geq 0$ on Ω), then for all $B(x,R) \subseteq \Omega$ and for all $r \in (0,R)$, the following holds:

$$u(x) \leq \frac{1}{4\pi r^2} \int_{\partial B(x,r)} u(y)\, dS(y).$$

In other words, the value of a subharmonic function at a point is bounded from above by the average of this function on a sphere centered at this point.

Solution:

a) We let:

$$g(r) := \frac{1}{4\pi r^2} \int_{\partial B(x,r)} u(y)\, dS(y).$$

If we use the change of variables $y = x + rz$, then the ball $B(x,r)$ in the y-variable goes to the ball $B(0,1)$ in the z-variable. Moreover, $dS(y) = r^2\,dS(z)$. Hence:

$$g(r) = \frac{1}{4\pi}\int_{\partial B(0,1)} u(x+rz)\,dS(z).$$

b) We differentiate under the integral sign and we use the Chain Rule to deduce that:

$$g'(r) = \frac{1}{4\pi}\int_{\partial B(0,1)} \nabla u(x+rz)\cdot z\,dS(z).$$

We recall that $z = \frac{y-x}{r}$, which is the outward pointing unit normal vector to $\partial B(x,r)$ at the point $y \in \partial B(x,r)$. We change variables back to y and we recall that: $dS(z) = \frac{1}{r^2}\,dS(y)$ to deduce that:

$$g'(r) = \frac{1}{4\pi r^2}\int_{\partial B(x,r)} \nabla u(y)\cdot \vec{n}(y)\,dS(y)$$

as was claimed.

c) We use the Divergence Theorem to deduce that:

$$g'(r) = \frac{1}{4\pi r^2}\int_{B(x,r)} \Delta u(y)\,dy.$$

Since $\Delta u = 0$, it follows that $g'(r) = 0$.

d) Since u is continuous, it follows that $\lim_{r\to 0} g(r) = u(x)$.

e) Combining parts c) and d), it follows that $g(r) = u(x)$ for all $r \in (0,R)$.

f) Suppose now that u is subharmonic, i.e. $\Delta u \geq 0$. We again define

$$g(r) := \frac{1}{4\pi^2}\int_{\partial B(x,r)} u(y)\,dS(y).$$

We argue as before to deduce that:

$$g'(r) = \frac{1}{4\pi r^2}\int_{B(x,r)} \Delta u(y)\,dy \geq 0.$$

Hence, in this case, g is an increasing function of r. Again, we know that, by continuity, $\lim_{r\to 0} g(r) = u(x)$. Hence, it follows that, for all $r \in (0,R)$, it holds that:

$$u(x) \leq g(r) = \frac{1}{4\pi^2}\int_{\partial B(x,r)} u(y)\,dS(y).\ \square$$

Remark: This approach has the advantage that it is computationally a bit simpler than the approach based on Poisson's formula. Moreover, as we note from part e), it can be applied in the context of subharmonic functions, which was not the case before. The drawback of this approach in comparison to the Poisson formula is the fact that it can only tell us the value of the function at the center of the ball whereas the Poisson formula tells us the value of the function at all points inside the ball.

Exercise 2. *(An application of the maximum principle for subharmonic functions)*
Suppose that $\Omega \subseteq \mathbb{R}^2$ is a bounded domain. Suppose that $v : \overline{\Omega} \to \mathbb{R}$ is continuous and suppose that, for some constant $C \in \mathbb{R}$:

$$v_{xx} + v_{yy} = C \text{ on } \Omega.$$

a) Show that the function $u = |\nabla v|^2$ is subharmonic on Ω.
b) Deduce that u achieves its maximum on $\partial\Omega$.

Solution:

a) We compute:

$$u_x = (v_x^2 + v_y^2)_x = 2v_x v_{xx} + 2v_y v_{yx}.$$
$$u_{xx} = (2v_x v_{xx} + 2v_y v_{yx})_x = 2v_{xx}^2 + 2v_x v_{xxx} + 2v_{yx}^2 + 2v_y v_{yxx}.$$
$$u_y = (v_x^2 + v_y^2)_y = 2v_x v_{xy} + 2v_y v_{yy}.$$
$$u_{yy} = (2v_x v_{xy} + 2v_y v_{yy})_y = 2v_{xy}^2 + 2v_x v_{xyy} + 2v_{yy}^2 + 2v_y v_{yyy}.$$

It follows that:

$$\Delta u = 2v_{xx}^2 + 4v_{xy}^2 + 2v_{yy}^2 + 2v_x(v_{xxx} + v_{xyy}) + 2v_y(v_{yxx} + v_{yyy})$$

Since $v_{xx} + v_{yy} = C$, we can differentiate both sides with respect to x and with respect to y respectively to deduce that:

$$v_{xxx} + v_{xyy} = v_{yxx} + v_{yyy} = 0.$$

In particular, it follows that:

$$\Delta u = 2v_{xx}^2 + 4v_{xy}^2 + 2v_{yy}^2 \geq 0.$$

Hence, u is subharmonic.

b) Since u is subharmonic, we can apply the result of Exercise 3 from Homework Assignment 6 to deduce that u achieves its maximum on $\partial\Omega$. □

Exercise 3. *(Poisson's equation on a ball in $\mathbb{R}^2$)*
Solve the Poisson's equation on the ball $B(0,1) \subseteq \mathbb{R}^2$:

$$\begin{cases} \Delta u = y, \ \text{on } B(0,1) \\ u = 1 \ \text{on } \partial B(0,1). \end{cases}$$

We will see that it is possible to use polar coordinates and separate variables to solve this problem. In general, we look for a solution of the form:

$$u(r,\theta) = \frac{1}{2}A_0(r) + \sum_{n=1}^{\infty} \Big\{ A_n(r)\cos(n\theta) + B_n(r)\sin(n\theta) \Big\}$$

where $A_0(r), A_1(r), B_1(r), \ldots$ are now functions of r. We are assuming that u is bounded near the origin.

a) Write the function y and the Laplace operator in polar coordinates (by the formula from class) and deduce that the functions $A_0, A_1, B_1, \ldots$ satisfy appropriate ODE initial value problems.

b) Solve these initial value problems and substitute the solutions into the formula for u. Write the answer as a function of x and y.
[HINT: When solving for B_1, we get the ODE $r^2 B_1'' + rB_1' - B_1 = r^3$. This ODE has a particular solution of the form $B_1(r) = Cr^3$. Use this fact to obtain the general solution to the ODE.]

Solution:

a) We note that $y = r\sin\theta$ in polar coordinates. Moreover, we recall the formula from class:

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}.$$

In particular, it follows that:

$$\Delta u = \Big(\frac{1}{2}A_0''(r) + \frac{1}{r}A_0'(r)\Big) + \sum_{n=1}^{\infty} \Big\{ \big(A_n'' + \frac{1}{r}A_n' - \frac{1}{r^2}n^2 A_n\big)\cos(n\theta) + \big(B_n'' + \frac{1}{r}B_n' - \frac{1}{r^2}n^2 B_n\big)\sin(n\theta) \Big\} = y\sin\theta$$

Consequently:

$$\frac{1}{2}A_0''(r) + \frac{1}{r}A_0'(r) = 0$$

$$A_n'' + \frac{1}{r}A_n' - \frac{1}{r^2}n^2 A_n = 0, \text{ for } n \geq 1.$$

$$B_1'' + \frac{1}{r}B_1' - \frac{1}{r^2}B_1 = r.$$

$$B_n'' + \frac{1}{r}B_n' - \frac{1}{r^2}n^2 B_n = 0, \text{ for } n \geq 2.$$

The boundary conditions imply that:

$$A_0(1) = 2.$$

$$A_n(1) = B_n(1) = 0, \text{ for } n \geq 1.$$

Thus, we obtain the following ODE initial value problems:

$$\begin{cases} r^2 A_0''(r) + rA_0'(r) = 0, \text{ for } r \in (0,1) \\ A_0(1) = 2. \end{cases}$$

For $n \geq 1$:

$$\begin{cases} r^2 A_n''(r) + rA_n'(r) - n^2 A_n(r) = 0, \text{ for } r \in (0,1) \\ A_n(0) = 0. \end{cases}$$

$$\begin{cases} r^2 B_1'' + rB_1(r) - B_1(r) = r^3, \text{ for } r \in (0,1) \\ B_1(0) = 0. \end{cases}$$

For $n \geq 2$:

$$\begin{cases} r^2 B_n'' + rB_n' - n^2 B_n = 0, \text{ for } r \in (0,1) \\ B_n(0) = 0. \end{cases}$$

b) From class, we know that A_0 has to be constant (there is no $\log r$ term since we are assuming that our solution is bounded near the origin). Hence:

$$A_0(r) = 2 \text{ for all } r \in (0,1).$$

Moreover, we know that $A_n(r) = C_n r^n$, for $n \geq 1$. Since $A_n(1) = 0$, it follows that, for all $n \geq 1$, we obtain:

$$A_n(r) = 0 \text{ for all } r \in (0,1).$$

Similarly, if $n \geq 2$, we deduce that:

$$B_n(r) = 0 \text{ for all } r \in (0,1).$$

We need to find $B_1(r)$. We look for a particular solution $B_{1,p}$ to the ODE of the form $B_{1,p}(r) = Cr^3$. Then:

$$r^2 B_{1,p}'' + rB_{1,p}' - B_{1,p} = 8Cr^3$$

which we assume equals r^3. Hence, it follows that a particular solution is given by:

$$B_{1,p} = \frac{1}{8}r^3.$$

The solution to the homogeneous equation $r^2 B_{1,h}'' + rB_{1,h}' - B_{1,h} = 0$ is given by:

$$B_{1,h}(r) = \lambda r$$

(we are looking only for solutions which are bounded near $r = 0$).

In particular, since $B_1(1) = 0$, it follows that $\lambda = -\frac{1}{8}$ and:

$$B_1(r) = \frac{1}{8}\big(r^3 - r\big).$$

Putting everything together, it follows that:

$$u(r,\theta) = 1 + \frac{1}{8}\big(r^3 - r\big)\sin\theta.$$

Since $y = r\sin\theta$, we can write this in Euclidean coordinates as:

$$u(x,y) = 1 + \frac{1}{8}(x^2 + y^2 - 1)\cdot y\ . \quad \square$$