MATH 425, HOMEWORK 7

This homework is due by noon on Friday, March 29. Please leave your assignment in my mailbox. There are three problems. Each problem is worth 10 points.

Exercise 1. *(An alternative derivation of the mean value property in 3D)*
Suppose that u is a harmonic function on a domain $\Omega \subseteq \mathbb{R}^3$ and suppose that $B(x,R) \subseteq \Omega$. We want to show that:

$$u(x) = \frac{1}{4\pi r^2} \int_{\partial B(x,r)} u(y) dS(y)$$

for all $r \in (0,R)$. This is the mean value property for harmonic functions in three dimensions. In class, we showed the analogous claim in two dimensions by using Poisson's formula. In this exercise, we outline how to give an alternative proof of the mean value property.

a) Define the function $g : (0,R) \to \mathbb{R}$ by:

$$g(r) := \frac{1}{4\pi r^2} \int_{\partial B(x,r)} u(y)\, dS(y).$$

By using a change of variables which takes $B(x,R)$ to $B(0,1)$, show that:

$$g(r) = \frac{1}{4\pi} \int_{\partial B(0,1)} u(x+rz)\, dS(z).$$

(In this way the domain of integration no longer depends on r.)

b) Show that:

$$g'(r) = \frac{1}{4\pi r^2} \int_{\partial B(x,r)} \nabla u(y) \cdot \vec{n}(y)\, dS(y)$$

where $\vec{n}(y)$ is the outward-pointing unit normal vector to $\partial B(x,r)$ at the point $y \in \partial B(x,r)$.
[HINT: Differentiate under the integral sign and then undo the change of variables in a).]

d) Use the Divergence Theorem and the Laplace equation to deduce that $g'(r) = 0$.

e) What is $\lim_{r\to 0} g(r)$? (Recall that u is smooth. In particular, it is continuous).

f) Conclude the proof of the mean value property.

g) [Extra Credit (2 points)] Show that if u is assumed to be continuous and subharmonic on Ω (i.e. $\Delta u \geq 0$ on Ω), then for all $B(x,R) \subseteq \Omega$ and for all $r \in (0,R)$, the following holds:

$$u(x) \leq \frac{1}{4\pi r^2} \int_{\partial B(x,r)} u(y)\, dS(y).$$

In other words, the value of a subharmonic function at a point is bounded from above by the average of this function on a sphere centered at this point.

Exercise 2. *(An application of the maximum principle for subharmonic functions)*
Suppose that $\Omega \subseteq \mathbb{R}^2$ is a bounded domain. Suppose that $v : \overline{\Omega} \to \mathbb{R}$ is continuous and suppose that, for some constant $C \in \mathbb{R}$:

$$v_{xx} + v_{yy} = C \text{ on } \Omega.$$

a) Show that the function $u = |\nabla v|^2$ *is subharmonic on* Ω.
b) Deduce that u *achieves its maximum on* $\partial\Omega$.

Exercise 3. *(Poisson's equation on a ball in* $\mathbb{R}^2$*)*
Solve the Poisson's equation on the ball $B(0,1) \subseteq \mathbb{R}^2$*:*

$$\begin{cases} \Delta u = y, \ \text{on } B(0,1) \\ u = 1 \ \text{on } \partial B(0,1). \end{cases}$$

We will see that it is possible to use polar coordinates and separate variables to solve this problem. In general, we look for a solution of the form:

$$u(r,\theta) = \frac{1}{2}A_0(r) + \sum_{n=1}^{\infty} \Big\{ A_n(r)\cos(n\theta) + B_n(r)\sin(n\theta) \Big\}$$

where $A_0(r), A_1(r), B_1(r), \ldots$ *are now functions of* r*. We are assuming that* u *is bounded near the origin.*

a) Write the function y *and the Laplace operator in polar coordinates (by the formula from class) and deduce that the functions* $A_0, A_1, B_1, \ldots$ *satisfy appropriate ODE initial value problems.*

b) Solve these initial value problems and substitute the solutions into the formula for u*. Write the answer as a function of* x *and* y*.*

[HINT: When solving for B_1*, we get the ODE* $r^2 B_1'' + rB_1' - B_1 = r^3$*. This ODE has a particular solution of the form* $B_1(r) = Cr^3$*. Use this fact to obtain the general solution to the ODE.]*