MATH 425, HOMEWORK 5

This homework is due by noon on Friday, March 15. Please leave your assignment in my mailbox. There are three problems. Each problem is worth 10 points.

Exercise 1. *(Uniqueness for the heat equation on $\mathbb{R}$)*

Suppose that the functions $u_1, u_2 : \mathbb{R}_x \times \mathbb{R}_t \to \mathbb{R}$ solve:

$$\begin{cases} \partial_t u_1 - k \cdot \partial_x^2 u_1 = 0, \ x \in \mathbb{R}, \ t > 0 \\ u_1(x,0) = \phi(x), \ x \in \mathbb{R} \end{cases}$$

and

$$\begin{cases} \partial_t u_2 - k \cdot \partial_x^2 u_2 = 0, \ x \in \mathbb{R}, \ t > 0 \\ u_2(x,0) = \phi(x), \ x \in \mathbb{R} \end{cases}$$

for some function $\phi : \mathbb{R} \to \mathbb{R}$.
Suppose furthermore that there exists constants $C, A > 0$ such that for all $x \in \mathbb{R}$ and for all $t > 0$, one has:

$$|u_1(x,t)| \le Ce^{Ax^2} \ \text{and} \ |u_2(x,t)| \le Ce^{Ax^2}.$$

Using the Global Maximum Principle (which was stated in class), show that:

$$u_1 = u_2.$$

(Here, one is allowed to use the result of the Global Maximum Principle, even though we didn't give the details of its proof in class.)

*This type of result is called **Conditional Uniqueness**. In other words, we know that solutions are unique in the class of objects satisfying some additional condition, which in this case is a bound of the type $|u(x,t)| \le Ce^{Ax^2}$.*

Exercise 2. *(The Global Maximum Principle in a special case)*

In this Exercise, we will give a proof of a special case of the Global Maximum Principle.
Suppose that

$$\begin{cases} u_t - k u_{xx} = 0, \ x \in \mathbb{R}, \ t > 0 \\ u(x,0) = \phi(x), \ x \in \mathbb{R} \end{cases}$$

for some bounded continuous function $\phi : \mathbb{R} \to \mathbb{R}$ which equals zero outside of $[0,1]$.
Suppose moreover that the solution u is bounded from above, i.e. that there exists $C > 0$ such that:

$$u(x,t) \le C \text{ for all } x \in \mathbb{R}, \ t > 0.$$

Let M_0 denote the maximum of ϕ (which exists by the assumptions on ϕ).

We want to prove that:

(1) $$u(x,t) \le M_0 \text{ for all } x \in \mathbb{R}, \ t > 0.$$

a) Fix $T > 0, L > 0$ and consider the rectangle: $Q_{L,T} := [-L,L]_x \times [0,T]_t$. Define on $Q_{T,L}$ the function:

$$w(x,t) := \frac{2C}{L^2} \cdot \left(\frac{x^2}{2} + kt \right) + M_0$$

Check that:
$$w_t - kw_{xx} = 0.$$

b) Explain how we can deduce that:
$$w \geq u \text{ on } Q_{L,T}.$$
[HINT: Recall the comparison results from the previous homework assignment.]

c) Fix (x_0, t_0). By using the result from part b), and by letting L tend to infinity, deduce the bound (1).

Exercise 3. *(Separation of variables for an inhomogeneous PDE)*
a) Solve the following boundary value problem by using the method of separation of variables:

$$\begin{cases} u_t - u_{xx} = \sin(2\pi x) + \sin(3\pi x), \ 0 < x < 1, \ t > 0 \\ u(x,0) = 0, \ 0 \leq x \leq 1 \\ u(0,t) = u(1,t) = 0, \ t > 0. \end{cases}$$

b) Solve the more general problem:

$$\begin{cases} v_t - v_{xx} = m \cdot v + \sin(2\pi x) + \sin(3\pi x), \ 0 < x < 1, \ t > 0 \\ v(x,0) = 0, \ 0 \leq x \leq 1 \\ v(0,t) = v(1,t) = 0, \ t > 0 \end{cases}$$

for $m \in \mathbb{R}$ a constant.

Full credit will be given for solving part b) when $m \neq 4\pi^2$ and $m \neq 9\pi^2$. Extra credit will be given for the solution of the problem in the cases $m = 4\pi^2$ and $m = 9\pi^2$.