MATH 425, HOMEWORK 4 SOLUTIONS

Exercise 1. *Consider the function $u(x,t) = 1 - x^2 - 2kt$, for $k > 0$ a constant.*

a) Verify that u is a solution to the heat equation.
b) Find the minimum and maximum of u on the closed rectangle $\{(x,t); 0 \leq x \leq 1,\ 0 \leq t \leq T\} = [0,1]_x \times [0,T]_t$ for a fixed $T > 0$, without using the maximum principle.
c) Find the minimum and maximum of u on $[0,1]_x \times [0,T]_t$ by using the maximum principle.

Solution:

a) We note that $u_t = -2k$ and $u_{xx} = -2$. Hence, $u_t = ku_{xx}$, so u solves the heat equation.

b) Let us first note that, for $(x,t) \in [0,1] \times [0,T]$, one has:

$$1 - x^2 - 2kt \leq 1$$

with equality if and only if $x = 0, t = 0$. $\Longrightarrow$ The maximum equals 1.

Moreover, for $(x,t) \in [0,1] \times [0,T]$, one also obtains:

$$1 - x^2 - 2kt \geq 1 - 1^2 - 2kT = -2kT.$$

Equality holds if and only if $x = 1, t = T$. $\Longrightarrow$ The minimum equals $-2kT$.

c) We will now use the Maximum Principle. It tells us that the maximum and minimum of u on the whole rectangle $[0,1] \times [0,T]$ are attained either on the bottom side (where $t = 0$) or on one of the lateral sides (where $x = 0$ or $x = 1$). Hence, we have to consider each of these sides separately.

- **t=0:** $u(x,0) = 1 - x^2 \Longrightarrow \max = 1$ (at $x = 0$) and $\min = 0$ (at $x = 1$).
- **x=0:** $u(0,t) = 1 - 2kt \Longrightarrow \max = 1$ (at $t = 0$) and $\min = 1 - 2kT$ (at $t = T$).
- **x=1:** $u(1,t) = -2kt \Longrightarrow \max = 0$ (at $t = 0$) and $\min = -2kT$ (at $t = T$).

Combining the previous cases, we deduce that the maximum of u on $[0,1] \times [0,T]$ equals 1, and it is achieved at $(x,t) = (0,0)$ and the minimum of u equals $-2kT$, and it is achieved at $(x,t) = (1,T)$. $\square$

Exercise 2. *Consider the initial value problem:*

$$\begin{cases} u_t - u_{xx} = 0, \text{ for } 0 < x < 1,\ t > 0 \\ u(x,0) = 4x(1-x), \text{ for } 0 \leq x \leq 1 \\ u(0,t) = 0,\ u(1,t) = 0, \text{ for } t > 0. \end{cases}$$

Show that:

$$u(x,t) = u(1-x,t),$$

for all $0 \leq x \leq 1, t \geq 0$.

Solution:

We define the function $v(x,t) := u(1-x,t)$. The function v is then defined on $[0,1]_x \times [0,+\infty)_t$. We use the Chain Rule to compute:

$$v_t(x,t) = u_t(1-x,t),\ v_x(x,t) = -u_x(1-x,t),\ v_{xx}(x,t) = u_{xx}(1-x,t).$$

Hence:

$$v_t(x,t) - v_{xx}(x,t) = u_t(1-x,t) - u_{xx}(1-x,t) = 0.$$

Thus, v solves the heat equation. We note that:

$$v(x,0) = u(1-x,0) = 4(1-x)x = 4x(1-x) = u(x,0) \text{ for all } 0 \le x \le 1.$$

Moreover, for all $t > 0$, we note that:

$$v(0,t) = v(1,t) = u(0,t) = u(1,t) = 0.$$

Consequently, v solves:

$$\begin{cases} v_t - v_{xx} = 0, \text{ for } 0 < x < 1, \ t > 0 \\ v(x,0) = 4x(1-x), \text{ for } 0 \le x \le 1 \\ v(0,t) = 0, \, v(1,t) = 0, \text{ for } t > 0. \end{cases}$$

We can now directly quote the uniqueness Theorem from class to see that $u = v$. $\square$

Exercise 3. *(A comparison principle)*
a) Suppose that u and v both solve the heat equation on $[0,L]_x \times (0,+\infty)_t$. Furthermore, suppose that $u \le v$ for $t = 0$, for $x = 0$, and for $x = L$. Show that:

$$u \le v \text{ on } [0,L]_x \times [0,+\infty)_t.$$

b) More generally, consider functions u and v which solve $u_t - ku_{xx} = f(x,t)$ and $v_t - kv_{xx} = g(x,t)$ on $[0,L]_x \times (0,+\infty)_t$. We assume moreover that $f \le g$ on $[0,L]_x \times (0,+\infty)_t$ and that $u \le v$ for $t = 0$, for $x = 0$, as well as for $x = L$. Show that $u \le v$ on $[0,L]_x \times [0,+\infty)_t$.

c) Suppose that the function v satisfies the inequality $v_t - v_{xx} \ge \sin x$ on $[0,\pi]_x \times (0,+\infty)$. Moreover, assume that: $v(0,t) \ge 0, v(\pi,t) \ge 0$ for all $t > 0$ and $v(x,0) \ge \sin x$ for all $0 \le x \le \pi$. Show that:

$$v(x,t) \ge (1 - e^{-t}) \sin x$$

on $[0,\pi]_x \times [0,+\infty)_t$.

Solution:

a) We take $w := u - v$, and we note that w solves the heat equation on $[0,L]_x \times (0,+\infty)_t$. In particular, if we fix a large finite time $T > 0$, the function w will solve the heat equation on $[0,L] \times (0,+\infty)_t$. By assumption, the function $w \le 0$ on the bottom and lateral sides of the rectangle $[0,L]_x \times [0,T]_t$. Thus, by the Maximum Principle, it follows that $w \le 0$ on $[0,L]_x \times [0,T]_t$. We can choose $T > 0$ to be arbitrarily large and we obtain that $w \le 0$ on $[0,L]_x \times [0,+\infty)_t$. In other words, $u \le v$ on $[0,L]_x \times [0,+\infty)_t$.

b) Throughout part b), let us fix a large finite time $T > 0$ and we consider the bounded rectangle $Q_T := [0,L]_x \times [0,T]_t$. We look at the function $w := u - v$ on Q_T. By construction, we know that:

$$w_t - kw_{xx} \le 0 \text{ on } [0,L]_t \times (0,T]_t. \tag{1}$$

Moreover, $w \le 0$ on the bottom side ($t = 0$) and on the lateral sides ($x = 0$ and $x = L$) of Q_T. The claim that we would like to show is that $\mathbf{w \le 0 \text{ on } Q_T}$.

Let us note that we can't directly apply the version of the Maximum Principle we proved in class to the (1) since this is an *inequality*. In other words, we don't necessarily know that w solves the heat equation. However, we will use the proof of the Maximum Principle from class and see that it still applies to the case of an inequality as in (1).

The idea is that, if w were to achieve its maximum at some point $(x_0, y_0) \in \tilde{Q}_T := (0,L)_x \times (0,T]_t$ (i.e. not on the bottom or lateral sides), then one would have: $w_t(x_0,t_0) \ge 0, w_{xx}(x_0,t_0) \le 0$, and so $w_t(x_0,t_0) - kw_{xx}(x_0,t_0) \ge 0$. This doesn't immediately give us a contradiction to (1) since one could have equality (which occurs in the case when $w_{xx}(x_0,t_0) = 0$).

We formalize this idea as in class by looking at an *approximation* of w, i.e. for $\epsilon > 0$, we define:

$$w^{\epsilon}(x,t) := w(x,t) + \epsilon x^2.$$

If we were to show that, for all $\epsilon > 0$:

$$w^{\epsilon} \leq \epsilon L^2 \text{ on } Q_T \tag{2}$$

Then, one would obtain that, for all $(x,t) \in Q_T$, one has:

$$w(x,t) \leq \epsilon(L^2 - x^2).$$

By letting $\epsilon \to 0$, we would obtain that $w \leq 0$ on Q_T, as claimed.

Hence, we reduce the claim to proving (2). We note that (2) follows if we show that the maximum of w^{ϵ} on Q_T can only be attained on the bottom and lateral sides.

We now argue by contradiction. Namely, we suppose that there exists $(x_0, t_0) \in Q_T$ at which w^{ϵ} attains its maximum on Q_T, but which doesn't lie on the bottom or lateral sides of Q_T. We need to consider two possibilities.

1) (x_0, t_0) lies in the interior of Q_T, i.e. $(x_0, t_0) \in (0, L)_x \times (0, T)_t$. In this case, we note that $w^{\epsilon}_t(x_0, t_0) = 0$ and $w^{\epsilon}_{xx}(x_0, t_0) \leq 0$. In particular, it follows that:

$$w^{\epsilon}_t(x_0,t_0) - k w^{\epsilon}_{xx}(x_0,t_0) \geq 0.$$

However, by construction of w^{ϵ}, we know that:

$$w^{\epsilon}_t = w, w^{\epsilon}_{xx} = w_{xx} + 2\epsilon$$

and so:

$$w^{\epsilon}_t - k w^{\epsilon}_{xx} = w_t - k w_{xx} - 2k\epsilon \leq -2k\epsilon < 0.$$

In this way, we obtain a contradiction.

2) (x_0, t_0) lies on the top side of Q_T, i.e. $0 < x_0 < L$ and $t_0 = T$. In this case, we know that $w^{\epsilon}_t(x_0, t_0) \geq 0$. Let us note that the derivative in t here is a derivative from the left. We still know that $w^{\epsilon}_{xx}(x_0, t_0) \leq 0$. Hence:

$$w^{\epsilon}_t(x_0,t_0) - k w^{\epsilon}_{xx}(x_0,t_0) \geq 0.$$

The argument now proceeds as in the previous case, and we obtain a contradiction, as before.

It follows that $\mathbf{w \leq 0}$ **on** $\mathbf{Q_T}$.

Since T is arbitrary, we can deduce that:

$$w \leq 0 \, \text{on} \, [0, L]_x \times [0, +\infty)_t.$$

In other words, we obtain that:

$$u \leq v \, \text{on} \, [0, L]_x \times [0, +\infty)_t.$$

c) Let us take $u(x,t) := (1 - e^{-t}) \sin x$.

We note that:

$$u_t(x,t) = e^{-t} \sin x \, \text{and} \, u_{xx}(x,t) = -(1 - e^{-t}) \sin x.$$

Hence,

$$u_t - u_{xx} = \sin x.$$

Moreover, we know that:

$$u(0,t) = u(\pi, t) \, \text{for all} \, t > 0.$$

Finally, since sin is non-negative on $[0, \pi]$, we note that, for all $x \in [0, \pi]$, one has:

$$u(x, 0) = 0 \leq \sin x = v(x, 0).$$

From part b), it follows that:

$$u \leq v \, \text{on} \, [0, \pi]_x \times [0, +\infty)_t.$$

In other words,

$$v(x,t) \geq (1 - e^{-t}) \sin x \,\text{on}\, [0,\pi]_x \times [0,+\infty)_t. \square$$