MATH 425, HOMEWORK 3 SOLUTIONS

Exercise 1. *(The differentiation property of the heat equation)*
In this exercise, we will use the fact that the derivative of a solution to the heat equation again solves the heat equation.

In particular, let us consider the following initial value problem:

$$\begin{cases} u_t - u_{xx} = 0, \text{ for } x \in \mathbb{R}\,, t > 0 \\ u(x,0) = x^2. \end{cases} \tag{1}$$

a) Let $v := u_{xxx}$. What initial value problem does v solve?
b) Use this observation to deduce that we can take $v = 0$.
c) What does this tell us about the form of u?
d) Use the latter expression to find a solution of (1).
e) Alternatively, solve for u using the explicit formula for the solution of the initial value problem for the heat equation.
f) Combine parts d) and e) to deduce the value of the integral $\int_{-\infty}^{+\infty} x^2 e^{-x^2}\,dx$.

(In part f), one is allowed to use the fact that the solutions obtained from part d) and part e) are identically equal without proof. We will study uniqueness of solutions to the heat equation later on in the class.)

Solution:

a) By the differentiation property, v also solves the heat equation. We note that $v_{xxx}(x,0) = 0$. Hence, v solves the initial value problem:

$$\begin{cases} v_t - v_{xx} = 0, \text{ for } x \in \mathbb{R}\,, t > 0 \\ v(x,0) = 0. \end{cases}$$

b) We note that the function $v = 0$ solves the initial value problem in part a).
c) From part b), we observe that we can look for a solution to (1) of the form:

$$u(x,t) = A(t) + B(t) \cdot x + C(t) \cdot x^2 \tag{2}$$

for some (differentiable) functions $A, B, C : \mathbb{R}_t^+ \to \mathbb{R}$ satisfying $A(0) = B(0) = 0, C(0) = 1$.
d) We note that, for u of the form (2), one has:

$$u_t - u_{xx} = (A'(t) - 2C(t)) + B'(t) \cdot x + C'(t) \cdot x^2$$

Hence, such a u solves the heat equation if and only if:

$$\begin{cases} A'(t) = 2C(t) \\ B'(t) = 0 \\ C'(t) = 0. \end{cases}$$

From the latter two conditions, it follows that B and C are constant. Since $B(0) = 0$ and $C(0) = 1$, we deduce that:

$$B(t) = 0 \text{ and } C(t) = 1.$$

We now use the first condition to deduce that:

$$A'(t) = 2C(t) = 2.$$

Since $A(0) = 0$, we conclude that $A(t) = 2t$. Putting all of this together, we obtain:

$$u(x,t) = 2t + x^2. \tag{3}$$

We readily check that the function u defined in (3) solves the initial value problem (1). Namely: $u_t = u_{xx} = 2$, hence $u_t - u_{xx} = 0$ and $u(x,0) = 0 + x^2 = x^2$.

e) We use the formula from class and recall that we are taking the diffusion coefficient to equal to 1, and hence u given by:

$$u(x,t) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{+\infty} e^{-\frac{(x-y)^2}{4t}} \cdot y^2 \, dy \tag{4}$$

solves (1).

f) Combining (3) and (4), and using the uniqueness of solutions to the heat equation (which we will assume in the problem without proof), it follows that, for all $x \in \mathbb{R}$ and $t > 0$, one has:

$$2t + x^2 = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{+\infty} e^{-\frac{(x-y)^2}{4t}} \cdot y^2 \, dy.$$

We substitute $x = 0$ and $t = \frac{1}{4}$, in which case $4t = 1$, and so:

$$\frac{1}{\sqrt{\pi}} \cdot \int_{-\infty}^{+\infty} y^2 \cdot e^{-y^2} \, dy = \frac{1}{2}.$$

Consequently,

$$\int_{-\infty}^{+\infty} x^2 e^{-x^2} \, dx = \frac{\sqrt{\pi}}{2}. \square$$

Exercise 2. *(The fundamental solution of the heat equation on $\mathbb{R}^2$)*
Recall that the function $S(x,t) = \frac{1}{\sqrt{4\pi kt}} e^{-\frac{x^2}{4kt}}$, defined for $x \in \mathbb{R}, t > 0$, is called ***the fundamental solution of the heat equation on*** $\mathbb{R}$.
a) Show that $S^{(2)}(x_1, x_2, t) := S(x_1, t) \cdot S(x_2, t)$ solves the heat equation on $\mathbb{R}^2$, i.e

$$S^{(2)}_t - k \cdot \Delta S^{(2)} = 0 \text{ on } \mathbb{R}^2_{(x_1,x_2)} \times \mathbb{R}^+_t.$$

b) Compute $\int_{\mathbb{R}^2} S^{(2)}(x,t) \, dx$, whenever $t > 0$.
c) Describe how the functions $S^{(2)}(x,t)$ look when we vary the parameter $t > 0$.
d) Consider the initial value problem:

$$\begin{cases} u_t - k \cdot \Delta u = 0, \text{ for } x = (x_1, x_2) \in \mathbb{R}^2, \, t > 0 \\ u(x,0) = \phi(x). \end{cases} \tag{5}$$

for ϕ a bounded and continuous function on $\mathbb{R}^2$.

Show that:

$$u(x,t) := \int_{\mathbb{R}^2} S^{(2)}(x-y,t) \phi(y) \, dy \tag{6}$$

solves the heat equation.
e) Show that the the function u defined in (6) satisfies the initial condition in the sense that, for any fixed $x \in \mathbb{R}^2$, one has:

$$\lim_{t \to 0+} u(x,t) = \phi(x).$$

Solution:

a) We compute, by using the Product Rule:

$$\frac{\partial}{\partial t} S^{(2)}(x_1, x_2, t) = \Big(\frac{\partial}{\partial t} S(x_1, t)\Big) \cdot S(x_2, t) \, + \, S(x_1, t) \cdot \Big(\frac{\partial}{\partial t} S(x_2, t)\Big).$$

Since S solves the one-dimensional heat equation with diffusion coefficient k, it follows that the above expression equals:

$$k \cdot \Big(\frac{\partial^2}{\partial x_1^2} S(x_1,t)\Big) \cdot S(x_2,t) \,+\, k \cdot S(x_1,t) \cdot \Big(\frac{\partial^2}{\partial x_2^2} S(x_2,t)\Big),$$

which, by the product rule again equals:

$$k \cdot \Delta S(x_1,t) \cdot S(x_2,t) = k \cdot S^{(2)}(x_1,x_2,t).$$

In other words,

$$S^{(2)}_t - k \cdot \Delta S^{(2)} = 0, \text{ on } \mathbb{R}^2_{(x_1,x_2)} \times \mathbb{R}^+_t.$$

b) We write $x = (x_1, x_2)$ and we note that, for all $t > 0$:

$$\int_{\mathbb{R}^2} S^{(2)}(x,t)\,dx = \int_{\mathbb{R}^2} S^{(2)}(x_1,x_2,t)\,dx_1 dx_2 =$$

$$= \Big(\int_{\mathbb{R}} S(x_1,t)\,dx_1\Big) \cdot \Big(\int_{\mathbb{R}} S(x_2,t)\,dx_2\Big) = 1 \cdot 1 = 1.$$

c) We write explicitly:

$$S^{(2)}(x,t) = \frac{1}{4\pi k t} e^{-\frac{x_1^2}{4kt} - \frac{x_2^2}{4kt}} = \frac{1}{4\pi t} e^{-\frac{|x|^2}{4kt}}.$$

Here $|x|^2 := x_1^2 + x_2^2$ is the square norm of the vector $x \in \mathbb{R}^2$. The 2D graphs of the functions $x \mapsto S^{(2)}(x,t)$ are radial and peaked at the origin. As t becomes smaller, they become more peaked and as t becomes larger, they become more spread out. The volume underneath each graph is equal to 1 by part b).

d) We can differentiate under the integral sign and deduce that the function u defined in (6) solves the heat equation:

$$u_t(x,t) - k \cdot \Delta u(x,t) = \int_{\mathbb{R}^2} \Big(\frac{\partial}{\partial t} - k \cdot \Delta\Big) S^{(2)}(x-y,t)\phi(y)\,dy = 0.$$

e) We argue similarly as in the one-dimensional case; since $\int_{\mathbb{R}^2} S^{(2)}(y,t)\,dy = 1$, we can write for fixed $x \in \mathbb{R}^2$ and $t > 0$:

$$u(x,t) - \phi(x) = \int_{\mathbb{R}^2} S^{(2)}(x-y,t)\phi(y)\,dy - \int_{\mathbb{R}^2} S^{(2)}(x-y)\phi(x)\,dy =$$

$$= \int_{\mathbb{R}^2} S^{(2)}(x-y,t) \cdot \Big(\phi(y) - \phi(x)\Big)\,dy.$$

It follows that:

$$\Big|u(x,t) - \phi(x)\Big| \leq \int_{\mathbb{R}^2} S^{(2)}(x-y,t) \cdot \Big|\phi(y) - \phi(x)\Big|\,dy. \tag{7}$$

Let $\epsilon > 0$ be given. By continuity of ϕ, we can find $\delta > 0$ such that for all $y \in \mathbb{R}^2$ with $|y - x| \leq \delta$, one has:

$$|\phi(y) - \phi(x)| \leq \epsilon. \tag{8}$$

We now write (7) as:

$$\Big|u(x,t) - \phi(x)\Big| \leq \int_{|y-x|\leq\delta} S^{(2)}(x-y,t) \cdot \Big|\phi(y) - \phi(x)\Big|\,dy + \int_{|y-x|>\delta} S^{(2)}(x-y,t) \cdot \Big|\phi(y) - \phi(x)\Big|\,dy. \tag{9}$$

We use (8) in order to deduce that the first term is:

$$\leq \epsilon \cdot \int_{|y-x|\leq\delta} S^{(2)}(x-y,t)\,dy$$

Since $S^{(2)} > 0$ ($S^{(2)}$ is a product of two positive functions), it follows that the above expression is:

$$\leq \epsilon \cdot \int_{\mathbb{R}^2} S^{(2)}(x-y,t)\,dy = \epsilon. \tag{10}$$

Let us now bound the second term in (9). We recall that, by assumption, the function ϕ is bounded. Hence, we can find $M > 0$ such that $|\phi(y)| \leq M$, for all $y \in \mathbb{R}^2$. In particular, the second term in (9) is bounded from above by:

$$2M \cdot \int_{|y-x|>\delta} S^{(2)}(x-y,t)\,dy = 2M \cdot \int_{|y-x|>\delta} \frac{1}{4\pi kt} \cdot e^{-\frac{|x-y|^2}{4kt}}\,dy =$$

$$= \begin{cases} w = \frac{y-x}{\sqrt{4kt}} \\ dw = \frac{1}{4kt}\,dy \end{cases} = \frac{2M}{\pi} \cdot \int_{|w|\geq\frac{\delta}{\sqrt{4kt}}} e^{-w^2}\,dw \to 0, \text{ as } t \to 0+.$$

In particular, we can find $t_\epsilon > 0$ such that for all $t \in (0, t_\epsilon)$, one has:

$$\int_{|y-x|>\delta} S^{(2)}(x-y,t) \cdot \Big|\phi(y) - \phi(x)\Big|\,dy \leq \epsilon. \tag{11}$$

Combining (9), (10) and (11), it follows that $|u(x,t) - \phi(x)| \leq 2\epsilon$ whenever $t \in (0, t_\epsilon)$. The time t_ϵ converges to zero as ϵ converges to zero. The claim follows when we let $\epsilon \to 0$. □

Exercise 3. *(The inhomogeneous heat equation with variable dissipation)*
In this exercise, we will find an explicit solution to the following initial value problem:

$$\begin{cases} u_t - k \cdot u_{xx} + bt^2 \cdot u = f(x,t), \text{ for } x \in \mathbb{R},\, t > 0 \\ u(x,0) = \phi(x). \end{cases} \tag{12}$$

Here, $b \in \mathbb{R}$ is a constant and $f = f(x,t)$ is a function.
a) Consider first the equation $w_t + bt^2 \cdot w = 0$ and solve it by using an integrating factor.
b) Using the insight from part a) and consider the function $v(x,t) := e^{\frac{bt^3}{3}} u(x,t)$, where u solves (12). *Which initial value problem does v solve?*
c) Use part b) in order to solve (12).

Solution:

a) We note that the integrating factor equals $e^{\frac{bt^3}{3}}$. When we multiply the equation for w with the integrating factor, we obtain:

$$\left(e^{\frac{bt^3}{3}} w\right)_t = 0.$$

As a result, the solution to the PDE for w is:

$$w(x,t) = F(x) e^{-\frac{bt^3}{3}}$$

for some function $F = F(x)$.

b) Using the insight of part a), we look at the function:

$$v(x,t) := e^{\frac{bt^3}{3}} \cdot u(x,t).$$

This transformation will help us transform the equation in (12) into an easier form (just as the integrating factor simplified the equation for w). In particular,

$$v_t(x,t) = e^{\frac{bt^3}{3}} \cdot u_t(x,t) + bt^2 e^{\frac{bt^3}{3}} \cdot u(x,t)$$

and

$$v_{xx}(x,t) = e^{\frac{bt^3}{3}} u_{xx}(x,t).$$

In particular,

$$v_t(x,t) - k \cdot v_{xx}(x,t) = e^{\frac{bt^3}{3}} \cdot \Big(u_t(x,t) - k \cdot u_{xx}(x,t) + bt^2 \cdot u(x,t) \Big) = e^{\frac{bt^3}{3}} \cdot f(x,t).$$

Furthermore, let us note that $v(x,0) = u(x,0) = \phi(x)$. Hence, the function v solves the initial value problem:

$$\begin{cases} v_t - k \cdot v_{xx} + bt^2 \cdot v = e^{\frac{bt^3}{3}} \cdot f(x,t), \text{ for } x \in \mathbb{R},\, t > 0 \\ v(x,0) = \phi(x). \end{cases} \tag{13}$$

In other words, it solves the inhomogeneous heat equation with a slightly different right-hand side. Namely, the inhomogeneous term $f(x,t)$ gets replaced by the inhomogeneous term $e^{\frac{bt^3}{3}} f(x,t)$. The initial data stays the same.

c) We use Duhamel's principle to solve the initial value problem (13). In particular, we note that:

$$v(x,t) = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{4\pi kt}} e^{-\frac{(x-y)^2}{4kt}} \phi(y)\, dy + \int_0^t \int_{-\infty}^{+\infty} \frac{1}{\sqrt{4\pi k(t-s)}} e^{-\frac{(x-y)^2}{4k(t-s)}} \cdot e^{\frac{bs^3}{3}} \cdot f(y,s)\, dyds.$$

Since $u(x,t) = e^{-\frac{bt^3}{3}} v(x,t)$, it follows that:

$$u(x,t) = e^{-\frac{bt^3}{3}} \cdot \Big(\int_{-\infty}^{+\infty} \frac{1}{\sqrt{4\pi kt}} e^{-\frac{(x-y)^2}{4kt}} \phi(y)\, dy + \int_0^t \int_{-\infty}^{+\infty} \frac{1}{\sqrt{4\pi k(t-s)}} e^{-\frac{(x-y)^2}{4k(t-s)}} \cdot e^{\frac{bs^3}{3}} \cdot f(y,s)\, dyds \Big). \square$$