MATH 425, HOMEWORK 3

This homework is due in class on Thursday, February 7. There are three problems. Each problem is worth 10 points.

Exercise 1. *(The differentiation property of the heat equation)*
In this exercise, we will use the fact that the derivative of a solution to the heat equation again solves the heat equation.

In particular, let us consider the following initial value problem:

$$\begin{cases} u_t - u_{xx} = 0, \text{ for } x \in \mathbb{R}, t > 0 \\ u(x,0) = x^2. \end{cases} \tag{1}$$

a) Let $v := u_{xxx}$. What initial value problem does v solve?
b) Use this observation to deduce that we can take $v = 0$.
c) What does this tell us about the form of u?
d) Use the latter expression to find a solution of (1).
e) Alternatively, solve for u using the explicit formula for the solution of the initial value problem for the heat equation.
f) Combine parts d) and e) to deduce the value of the integral $\int_{-\infty}^{+\infty} x^2 e^{-x^2}\, dx$.

(In part f), one is allowed to use the fact that the solutions obtained from part d) and part e) are identically equal without proof. We will study uniqueness of solutions to the heat equation later on in the class.)

Exercise 2. *(The fundamental solution of the heat equation on $\mathbb{R}^2$)*
*Recall that the function $S(x,t) = \frac{1}{\sqrt{4\pi k t}} e^{-\frac{x^2}{4kt}}$, defined for $x \in \mathbb{R}, t > 0$, is called **the fundamental solution of the heat equation on** $\mathbb{R}$.*
a) Show that $S^{(2)}(x_1, x_2, t) := S(x_1,t) \cdot S(x_2,t)$ solves the heat equation on $\mathbb{R}^2$, i.e

$$S_t^{(2)} - k \cdot \Delta S^{(2)} = 0 \text{ on } \mathbb{R}^2_{(x_1,x_2)} \times \mathbb{R}_t^+.$$

b) Compute $\int_{\mathbb{R}^2} S^{(2)}(x,t)\, dx$, whenever $t > 0$.
c) Describe how the functions $S^{(2)}(x,t)$ look when we vary the parameter $t > 0$.
d) Consider the initial value problem:

$$\begin{cases} u_t - k \cdot \Delta u = 0, \text{ for } x = (x_1, x_2) \in \mathbb{R}^2,\ t > 0 \\ u(x,0) = \phi(x). \end{cases} \tag{2}$$

for ϕ a bounded and continuous function on $\mathbb{R}^2$.

Show that:

$$u(x,t) := \int_{\mathbb{R}^2} S^{(2)}(x-y,t)\phi(y)\, dy \tag{3}$$

solves the heat equation.
e) Show that the the function u defined in (3) satisfies the initial condition in the sense that, for any fixed $x \in \mathbb{R}^2$, one has:

$$\lim_{t \to 0+} u(x,t) = \phi(x).$$

(One more problem on the other side.)

Exercise 3. *(The inhomogeneous heat equation with variable dissipation)*
In this exercise, we will find an explicit solution to the following initial value problem:

$$\begin{cases} u_t - k \cdot u_{xx} + bt^2 \cdot u = f(x,t), \text{ for } x \in \mathbb{R},\, t > 0 \\ u(x,0) = \phi(x). \end{cases} \tag{4}$$

Here, $b \in \mathbb{R}$ *is a constant and* $f = f(x,t)$ *is a function.*
a) Consider first the equation $w_t + bt^2 \cdot w = 0$ *and solve it by using an integrating factor.*
b) Using the insight from part a) and consider the function $v(x,t) := e^{\frac{bt^3}{3}} u(x,t)$*, where* u *solves* (4)*.*
Which initial value problem does v *solve?*
c) Use part b) in order to solve (4)*.*