MATH 425, HOMEWORK 2 SOLUTIONS

Exercise 1. *(The role of the diffusion coefficient)*
In this exercise, we will see how to justify the fact that we can set the diffusion coefficient in the heat equation to equal 1*. Furthermore, we will see the importance of the sign of the diffusion coefficient.*

Suppose that we are looking at a function $u : \mathbb{R}^n_x \times \mathbb{R}^+_t \to \mathbb{R}$ *such that:*

$$\begin{cases} u_t - k \cdot \Delta u = 0 \\ u\big|_{t=0} = \Phi. \end{cases} \tag{1}$$

for some constant $k > 0$ *and for some function* $\Phi = \Phi(x)$*.*
a) Suppose that u *is a solution of* (1)*. By appropriately choosing the constants* a *and* b*, construct a function* $v : \mathbb{R}^n_x \times \mathbb{R}^+_t$ *of the form* $v(x,t) = u(ax, bt)$ *which solves the equation* $v_t - \Delta v = 0$ *on* $\mathbb{R}^n_x \times \mathbb{R}^+_t$*.*
b) Express $v\big|_{t=0}$ *you obtained this way in terms of* Φ*.*
c) Explain briefly why this type of transformation can't give us a solution of the equation $w_t + \Delta w = 0$ *on* $\mathbb{R}^n_x \times \mathbb{R}^+_t$*.*
d) Take $w(x,t) = u(x,-t)$*, for* u *a solution of* (1)*. On what set is* w *defined?*
e) Show that the function w *defined in part d) solves* $w_t + k \cdot \Delta w = 0$ *on its domain of definition.*

Solution:

a) We let $v(x,t) := u(ax, bt)$ as in the problem. We notice that, in order for v to be defined on $\mathbb{R}^n_x \times \mathbb{R}^+_t$, we need to take $b > 0$. In other words, we don't want to change the sign of the time parameter. We now use the Chain Rule to compute:

$$v_t(x,t) = b \cdot u_t(ax, bt)$$

and

$$\Delta v(x,t) = a^2 \cdot \Delta u(ax, bt).$$

Consequently:

$$v_t(x,t) - \Delta v(x,t) = b \cdot u_t(ax,bt) - a^2 \cdot \Delta u(ax,bt) = b \cdot \Big(u_t(ax,bt) - \frac{a^2}{b} \cdot \Delta u(ax,bt)\Big) \tag{2}$$

This step is justified since we are assuming that $b \neq 0$. It follows that v will then solve the heat equation with diffusion coefficient equal to 1 if and only if:

$$k = \frac{a^2}{b}. \tag{3}$$

b) Let us note that, under the above dilation, the initial data Φ is transformed to:

$$v(x,0) = u(ax,0) = \Phi(ax).$$

c) We note that, by (2), under the transformation given in the problem, the diffusion coefficient equals $\frac{a^2}{b}$. Since we are considering a domain where the t parameter is positive, we must take $b > 0$, as we noted above. Hence, the diffusion constant under this transformation can't be negative. In particular, in this way, we can't construct a solution to $w_t + \Delta w = 0$ on $\mathbb{R}^n_x \times \mathbb{R}^+_t$.

d) The function $w(x,t) = u(x,-t)$ is defined on $\Big(\mathbb{R}^n_x \times \mathbb{R}^-_t\Big) \cup \Big(\mathbb{R}^n_x \times \{t = 0\}\Big)$. We note that the domain of the initial data doesn't change under the reflection in time.

e) By using the Chain Rule, we note that $w_t(x,t) = -u_t(x,-t)$. Furthermore, $\Delta w(x,t) = \Delta u(x,-t)$. Consequently, it follows that: $w_t + \Delta w = 0$ on $\mathbb{R}^n_x \times \mathbb{R}^+_t$. Strictly speaking, we can only say that the function w solves the backwards heat equation (i.e. heat equation with a negative diffusion coefficient) on the set $\mathbb{R}^n_x \times \mathbb{R}^-_t$. If we are taking one-sided derivatives in the time variable at $t = 0$, we can then say that it solves the PDE on the whole domain of its definition, i.e. on $\Big(\mathbb{R}^n_x \times \mathbb{R}^-_t\Big) \cup \Big(\mathbb{R}^n_x \times \{t = 0\}\Big)$. $\square$

Exercise 2. *(The heat equation with convection)*
a) Suppose that we know that the solution to the general heat equation initial value problem on $\mathbb{R}$:

$$\begin{cases} u_t - k \cdot u_{xx} = 0, \text{ for } x \in \mathbb{R},\ t > 0 \\ u(x,0) = \phi(x) \end{cases}$$

is given by:

$$u(x,t) = \int_{-\infty}^{+\infty} S(x-y,t)\phi(y)dy \tag{4}$$

for an explicitly determined function $S(x,t)$. (The fact that this is indeed the case will be shown in class on Tuesday, January 29.)
Assuming the formula (4), solve the initial value problem for the heat equation with convection:

$$\begin{cases} u_t - k \cdot u_{xx} + V \cdot u_x = 0, \text{ for } x \in \mathbb{R},\ t > 0 \\ u(x,0) = \psi(x). \end{cases}$$

Here, we are assuming that V is a real constant. The answer should be given in terms of an integral involving the function S.

(HINT: It is a good idea to use a "moving frame", i.e. to look at the function $\tilde{u}(x,t) := u(x+Vt,t)$. In the new coordinate system, the x coordinate "moves" with time at speed V.)
b) Give a brief explanation of what sort of physical phenomenon can be modeled by the equation in part a). (It is not necessary to derive the equation here; just give a one sentence description).

Solution:

a) Let us consider the function $\tilde{u}(x,t) := u(x+Vt,t)$. By using the Chain Rule, it follows that:

$$\tilde{u}_t(x,t) = V \cdot u_x(x+Vt,t) + u_t(x+Vt,t)$$

and:

$$\tilde{u}_{xx}(x,t) = u_{xx}(x+Vt,t).$$

Consequently:

$$\tilde{u}_t(x,t) - k \cdot \tilde{u}_{xx}(x,t) = u_t(x+Vt,t) - k \cdot \Delta u(x+Vt,t) + V \cdot u_x(x+Vt,t)$$

which equals zero for $x \in \mathbb{R}^n, t \in \mathbb{R}^+$, provided that u is a solution to the heat equation with convection. In other words, by using the heat equation with convection, we can build up a solution to the regular heat equation. Furthermore, we observe that:

$$u(x,t) = \tilde{u}(x-Vt,t). \tag{5}$$

Hence, we can reverse the procedure. Finally, let us note that $\tilde{u}(x,0) = u(x,0)$. We are thus led to study:

$$(6) \qquad \begin{cases} \tilde{u}_t - k \cdot \tilde{u}_{xx} = 0, \text{ for } x \in \mathbb{R}, \ t > 0 \\ \tilde{u}|_{t=0} = \psi(x) \end{cases}$$

By using the formula from class, we know that:

$$\tilde{u}(x,t) = \int_{-\infty}^{+\infty} S(x-y,t)\psi(y)dy.$$

By recalling (5), it follows that:

$$u(x,t) = \tilde{u}(x - Vt, t) = \int_{-\infty}^{+\infty} S(x - Vt - y, t)\psi(y)dy.$$

b) A possible model would be the diffusion of ink in a pipe, where the ink is additionally being transported at speed V to the right [1]. $\square$

Exercise 3. *(A consequence of invariance under dilations)*
Suppose that $Q : \mathbb{R}_x \times \mathbb{R}_t^+ \to \mathbb{R}$ is a function. For $a > 0$, we define the new function $Q^{(a)} : \mathbb{R}_x \times \mathbb{R}_t^+ \to \mathbb{R}$ by:

$$Q^{(a)}(x,t) := Q(\sqrt{a} \cdot x, a \cdot t).$$

Suppose that, for all $a > 0$, and for all $x \in \mathbb{R}, t > 0$, one has:

$$Q^{(a)}(x,t) = Q(x,t).$$

Show that, for all $C > 0$, the function Q is constant along the segment of the parabola given by:

$$x = C\sqrt{t};\ t > 0.$$

This exercise justifies our guess that $Q(x,t)$ is a function of $\frac{x}{\sqrt{t}}$.

Solution:

Let $C > 0$ be given. We want to show that the function Q is constant along the segment of the parabola given by: $x = C\sqrt{t}$ for $t > 0$. In other words, given $t_1, t_2 > 0$, we want to show that:

$$(7) \qquad Q(C\sqrt{t_1}, t_1) = Q(C\sqrt{t_2}, t_2).$$

Let us take $a = \frac{t_2}{t_1} > 0$. We note that:

$$\sqrt{a} \cdot C\sqrt{t_1} = C\sqrt{t_2}$$

and hence:

$$Q^{(a)}(C\sqrt{t_1}, t_1) = Q(\sqrt{a} \cdot C\sqrt{t_1}, a \cdot t_1) = Q(C\sqrt{t_2}, t_2).$$

Since, by assumption, $Q^{(a)} = Q$, it follows that:

$$Q(C\sqrt{t_1}, t_1) = Q(C\sqrt{t_2}, t_2). \square$$

Second Solution: We note that, for $x \in \mathbb{R}, t > 0$, one has:

$$Q(x,t) = Q(\sqrt{t} \cdot \frac{x}{\sqrt{t}}, t \cdot 1) = Q^{(t)}(\frac{x}{\sqrt{t}}, 1)$$

Since $Q^{(t)} = Q$, this expression equals:

$$Q(\frac{x}{\sqrt{t}}, 1).$$

[1] The way to see that the transport goes to the right is to see that the function u, i.e. the concentration of ink at point x at time t, takes the form $u(x,t) = \tilde{u}(x - vt, t)$.

The latter is a expression is a function of $\frac{x}{\sqrt{t}}$, namely, we can write it as $F(\frac{x}{\sqrt{t}})$, where $F(y) := Q(y,1)$. $\square$

Exercise 4. *(The Gaussian integral)*
In this exercise, we will summarize some important properties of a specific definite integral which we will need to use in order to study the heat equation on $\mathbb{R}$.
a) Show that:

$$\int_0^{+\infty} e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}.$$

(HINT: Look at the product $\int_0^{+\infty} e^{-x^2}dx \int_0^{+\infty} e^{-y^2}dy$ and apply polar coordinates.)
b) Deduce that: $\int_{-\infty}^0 e^{-x^2}dx = \frac{\sqrt{\pi}}{2}$ and hence: $\int_{-\infty}^{+\infty} e^{-x^2}dx = \sqrt{\pi}$.
c) Use a change of variables in order to conclude that: $\int_{-\infty}^{+\infty} S(x,t)dx = 1$ for the function $S(x,t) = \frac{1}{\sqrt{4\pi kt}}e^{-\frac{|x|^2}{4kt}}$, defined for $x \in \mathbb{R}$ and $t > 0$. Here, we are assuming that $k > 0$ is a given constant.

Solution:

a) We note that:

$$\Big(\int_0^{+\infty} e^{-x^2}\,dx\Big)^2 = \Big(\int_0^{+\infty} e^{-x^2}\,dx\Big)\cdot\Big(\int_0^{+\infty} e^{-y^2}\,dy\Big) = \int_0^{+\infty}\int_0^{+\infty} e^{-x^2-y^2}dxdy$$

which in polar coordinates equals:

$$\int_0^{\frac{\pi}{2}}\int_0^{+\infty} e^{-r^2}rdrd\theta = \frac{\pi}{2}\Big(-\frac{1}{2}e^{-r^2}\Big)\Big|_{r=0}^{r=+\infty} = \frac{\pi}{4}$$

Hence

$$\Big(\int_0^{+\infty} e^{-x^2}dx\Big)^2 = \frac{\pi}{4}$$

from where it follows that:

$$\int_0^{+\infty} e^{-x^2}dx = \frac{\sqrt{\pi}}{2}$$

as was claimed.

b) We note that $\int_{-\infty}^0 e^{-x^2}\,dx = \int_0^{+\infty} e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}$, and hence $\int_{-\infty}^{+\infty} e^{-x^2}\,dx = \sqrt{\pi}$.

c) We observe that:

$$\int_{-\infty}^{+\infty} S(x,t)\,dx = \int_{-\infty}^{+\infty}\frac{1}{\sqrt{4\pi kt}}e^{-\frac{|x|^2}{4kt}}\,dx$$

We change variables as $y = \frac{x}{\sqrt{4kt}}$. Hence $dy = \frac{1}{\sqrt{4kt}}dx$, and the integral equals:

$$\int_{-\infty}^{+\infty}\frac{1}{\sqrt{4\pi kt}}e^{-|y|^2}\cdot\sqrt{4kt}\,dy = \frac{1}{\sqrt{\pi}}\cdot\int_{-\infty}^{+\infty} e^{-y^2}\,dy = 1. \square$$

Alternative approach for part c) (without using a change of variables)
We can also differentiate under the integral sign: Namely:

$$\frac{d}{dt}\int_{-\infty}^{+\infty} S(x,t)\,dx = \int_{-\infty}^{+\infty}\frac{\partial}{\partial t}S(x,t)\,dx.$$

Since S solves the heat equation $S_t - k\cdot S_{xx} = 0$, this expression equals:

$$\int_{-\infty}^{+\infty} k\cdot\frac{\partial^2}{\partial x^2}S(x,t)\,dx = k\cdot\frac{\partial}{\partial x}S(x,t)\Big|_{x=-\infty}^{x=+\infty} = 0$$

by using the Fundamental Theorem of Calculus in the x variable and the fact that $\frac{\partial}{\partial x}S(x,t)$ vanishes at $x = \pm\infty$. $\square$