MATH 425, HOMEWORK 1, SOLUTIONS

Exercise 1. *We recall from class that an operator $\mathcal{L}$ acting on functions is said to be **linear** if for all functions u, v and for all scalars a, b, one has $\mathcal{L}(au + bv) = a \cdot \mathcal{L}u + b \cdot \mathcal{L}v$.*

Which of the following operators are linear?
a) $\mathcal{L}u = u_{xx} + u_{xy}$.
b) $\mathcal{L}u = u_t + uu_x$.
c) $\mathcal{L}u = \sin(x^2y)u_x + e^{xy^2}u_y$.
d) $\mathcal{L}u = u_x + u_y + 1$.
e) $\mathcal{L}u = u_{xx} + \sin(u)$.
Give a brief justification for each answer.

Solution:

a) YES. Partial differentiation is a linear operation, so the sum of two partial derivatives of u depends linearly on u.
b) NO. The operation $u \mapsto uu_x$ is nonlinear, whereas $u \mapsto u_t$ is, so the resulting map is nonlinear. For example, given a function u, we note that $\mathcal{L}(2u) = (2u)_t + (2u)(2u)_x = 2u_t + 4uu_x \neq 2u_t + 2uu_x = 2\mathcal{L}(u)$, i.e. $\mathcal{L}(2u) \neq 2\mathcal{L}(u)$.
c) YES. As in part a), we recall that partial differentiation is a linear operation. Taking coefficients which don't depend on u (in particular, $\sin(x^2y)$ and e^{xy^2} doesn't affect the linearity of the map.
d) NO. This map is not linear due to the presence of the constant factor. In particular, we note that $\mathcal{L}(0) = 1 \neq 0$. We recall that for a linear map T, it is always the case that $T(0) = 0$, since $T(0) = T(0 \cdot 0) = 0 \cdot T(0) = 0$.
e) NO. We use analogous reasoning as in part b). Namely, we note that $u \mapsto u_{xx}$ is linear and that $u \mapsto \sin(u)$ is not linear. In order to see the last claim, we can see, for instance that: $\sin(2u) \neq 2\sin(u)$. $\square$

In the following exercises, u is assumed to be a function of two variables.

Exercise 2. *(Strauss, Exercise 1.2.1.)*
Solve the first order PDE: $2u_t + 3u_x = 0$, *with the auxiliary condition* $u = \sin x$ *when* $t = 0$.

Solution:

We know from class that $u(x,t) = f(2x - 3t)$ for some (differentiable) function f. We take $t = 0$ to deduce that $u(x, 0) = f(2x) = \sin x$. In particular, it follows that $f(x) = \sin(\frac{x}{2})$. Consequently:

$$u(x,t) = \sin\Big(x - \frac{3}{2}t\Big).$$

The answer is immediately checked. $\square$

Exercise 3. *(Strauss, Exercise 1.2.3.)*
Solve the equation: $(1 + x^2)u_x + u_y = 0$. *Describe its characteristic curves.*

Solution:

We use the method of characteristics. Let us first rewrite the equation as:

$$u_x + \frac{1}{1+x^2}u_y = 0.$$

The characteristic ODE then becomes:

$$\frac{dy}{dx} = \frac{1}{1+x^2}$$

which, by Separation of Variables can be solved as:

$$y = \arctan x + C.$$

for some constant C. It follows that:

$$u(x,y) = f(y - \arctan x)$$

for some (differentiable) function f. Again, the answer is immediately checked. Moreover, the characteristic curves are the translates of the graph of the function $\arctan x$. □

Exercise 4. *(Strauss, Exercise 1.2.6.)*
a) Solve the equation: $yu_x + xu_y = 0$, *with the condition* $u(0,y) = e^{-y^2}$.
b) In which region of the xy-plane is the solution uniquely determined?

Solution:

a) We will apply the method of characteristics. We rewrite the PDE as:

$$u_x + \frac{x}{y} u_y = 0.$$

One then needs to solve:

$$\frac{dy}{dx} = \frac{x}{y}.$$

We can separate variables to deduce:

$$x\,dx = y\,dy$$

It follows that the characteristic curves are given are given by the connected components of:

$$x^2 - y^2 = C. \tag{1}$$

for $C \in \mathbb{R}$.

Let us first solve the problem in full generality and we later substitute the value of $u(0,y)$ (solving the problem directly also counts for full credit). One has to be a bit careful here; for $C \neq 0$, equation (1) gives us two segments of a hyperbola (so not one connected curve), and for $C = 0$, it gives us the union of the lines $y = x$ and $y = -x$. In any case, by the method of characteristics, **the function** u **will be constant on each of the connected components of these curves**. It follows that:

$$u(x,y) = \begin{cases} C, \text{ if } y = \pm x \\ g_1(x^2-y^2), \text{ if } x^2-y^2<0 \text{ and } y>0 \textbf{ (Upwards facing hyperbolic segments)} \\ g_2(x^2-y^2), \text{ if } x^2-y^2<0 \text{ and } y<0 \textbf{ (Downwards facing hyperbolic segments)} \\ h_1(x^2-y^2), \text{ if } x^2-y^2>0 \text{ and } x>0 \textbf{ (Rightwards facing hyperbolic segments)} \\ h_2(x^2-y^2), \text{ if } x^2-y^2>0 \text{ and } x<0 \textbf{ (Leftwards facing hyperbolic segments)} \end{cases}$$

Strictly speaking, we must choose the constant C and the functions g_1, g_2, h_1, h_2 in such a way that the resulting function u is differentiable. What is important to notice is the fact that the functions g_1 and g_2 are mutually independent. The same holds for the functions h_1 and h_2.
Now, we go back to the specific example where $u(0,y) = e^{-y^2}$. We note that $(0, y_0)$ is the intersection of the y axis with the set $x^2 - y^2 = -y_0^2$ in the half-plane where y has the same sign as y_0 (if $y_0 = 0$, this point is just $(0,0)$). Using this observation, the previous case-by-case formula for u, and the assumption that $u(0,y) = e^{-y^2}$, it follows that: $g_1(x) = x, g_2(x) = x$ and $C = 1$. In particular, we deduce that:

$$u(x,y) = \begin{cases} e^{x^2-y^2}, \text{ if } x^2-y^2 \leq 0. \\ h_1(x^2-y^2), \text{ if } x^2-y^2>0 \text{ and } x>0 \textbf{ (Rightwards facing hyperbolic segments)} \\ h_2(x^2-y^2), \text{ if } x^2-y^2>0 \text{ and } x<0 \textbf{ (Leftwards facing hyperbolic segments)} \end{cases}$$

Again, we need to choose the functions h_1 and h_2 in such a way that the function u is differentiable.
b) Since the value of u is given on the y-axis, it follows that the solution is uniquely determined along the characteristic curves which intersect the y-axis. These includes the upwards and downwards facing hyperbolic segments as well as the union of the lines $y = x$ and $y = -x$. Hence, the solution u is uniquely determined on the set where $x^2 - y^2 \leq 0$. In our previous notation, this means that we can determine the constant C and the functions g_1 and g_2, but the functions h_1 and h_2 cannot be determined from the given data. It is important to remark that the fact that the functions g_1 and g_2 are equal in our example is not the case for arbitrary $u(0, y)$. In the specific example it is due to the fact that $u(0, y) = u(0, -y)$, i.e. that the function $y \mapsto u(0, y)$ is even. $\square$

Exercise 5. *(Strauss, Exercise 1.2.11.)*
Use the coordinate method in order to solve the equation:

$$u_x + 2u_y + (2x - y)u = 2x^2 + 3xy - 2y^2.$$

Solution:

Let us take:

$$\begin{cases} x' = x + 2y \\ y' = 2x - y. \end{cases}$$

By the Chain Rule, we then obtain:

$$\begin{cases} u_x = u_{x'} \cdot \frac{\partial x'}{\partial x} + u_{y'} \cdot \frac{\partial y'}{\partial x} = u_{x'} + 2u_{y'} \\ u_y = u_{x'} \cdot \frac{\partial x'}{\partial y} + u_{y'} \cdot \frac{\partial y'}{\partial y} = 2u_{x'} - u_{y'}. \end{cases}$$

It follows that:

$$u_x + 2u_y = (u_{x'} + 2u_{y'}) + 2(2u_{x'} - u_{y'}) = 5u_{x'}.$$

Furthermore, we note that we can factorize the right-hand side of the original PDE (in the (x, y) variables):

$$2x^2 + 3xy - 2y^2 = 2x^2 + 4xy - xy - 2y^2 = 2x(x + 2y) - y(x + 2y) = (2x - y) \cdot (x + 2y) = x' \cdot y'.$$

From the preceding, it follows that we can rewrite the PDE in the (x', y') coordinates as:

$$5u_{x'} + y'u = x' \cdot y'$$

i.e:

$$u_{x'} + \frac{1}{5}y'u = \frac{1}{5}x' \cdot y'.$$

If we fix y', we can think of the above equation as a first order ODE in x'. We note that this equation can then be solved by using the integrating factor given by $e^{\frac{1}{5}x' \cdot y'}$. When we multiply the above PDE with the integrating factor, we obtain:

$$\left(e^{\frac{1}{5}x' \cdot y'}u\right)_{x'} = \frac{1}{5}x' \cdot y' \cdot e^{\frac{1}{5}x' \cdot y'}$$

Consequently,

$$e^{\frac{1}{5}x' \cdot y'}u(x', y') = F(y') + \int_0^{x'} \frac{1}{5}t \cdot y' \cdot e^{\frac{1}{5}t \cdot y'}dt. \tag{2}$$

for some function $F = F(y')$. We note that:

$$\int_0^x a \cdot te^{a \cdot t}dt = \begin{cases} u = a \cdot t, & du = adt \\ dv = e^{at}dt, & v = \frac{1}{a}e^{at} \end{cases}$$

$$= xe^{a \cdot x} - \int_0^x e^{a \cdot t}dt =$$

$$= xe^{a\cdot x} - \frac{1}{a}e^{a\cdot x} + \frac{1}{a}. \tag{3}$$

We substitute (3) into (2) with $a = \frac{1}{5}y'$ to deduce that:

$$e^{\frac{1}{5}x'\cdot y'}u(x',y') = F(y') + x'\cdot e^{\frac{1}{5}x'\cdot y'} - \frac{5}{y'}\cdot e^{\frac{1}{5}x'\cdot y'} + \frac{5}{y'}$$

It follows that:

$$u(x',y') = x' - \frac{5}{y'} + e^{-\frac{1}{5}x'\cdot y'}\cdot f(y')$$

for the function $f(y') = F(y') + \frac{5}{y'}$.

Finally, we can change back to the original coordinates (x, y) to deduce that:

$$u(x,y) = \big(x+2y\big) - \frac{5}{2x-y} + e^{\frac{-2x^2-3xy+2y^2}{5}}\cdot f(2x-y)$$

for some function $f : \mathbb{R} \to \mathbb{R}$. $\square$