MATH 425, FINAL EXAM. APRIL 29, 2013.

No books, notes or calculators are allowed during the test. There are five exercises. Each exercise is worth 50 points. In your solutions, you should prove the claims that you are using, unless it is specified otherwise in the statement of the problem. Good Luck!

Exercise 1. *a) The operator $\mathcal{L}_1$ is defined on smooth functions of (x, y) by:*

$$\mathcal{L}_1(u) := \arctan(xy) \cdot u_{xx} + \sin(x^2y^2) \cdot u_{yy}.$$

Is the operator $\mathcal{L}_1$ linear? Prove your answer.

b) Does the answer change if we replace the operator $\mathcal{L}_1$ by the operator $\mathcal{L}_2$, which is given by:

$$\mathcal{L}_2(u) := u_{xx} + e^u \ ?$$

c) Find the general solution of the PDE $u_x + x^2u_y = 0$ by using the method of characteristics. Check that your solution solves the PDE. You don't need to show that these are all of the solutions.

Exercise 2. *In this exercise, we would like to find a solution to the following initial value problem:*

$$\begin{cases} u_t - u_{xx} = 0, \ \textit{for } x \in \mathbb{R}\,, t > 0 \\ u(x, 0) = x^2, \ \textit{for } x \in \mathbb{R}. \end{cases} \tag{1}$$

a) Let $v := u_{xxx}$. What initial value problem does v solve?

b) Use this observation to deduce that we can take $v = 0$ to be a solution of the initial value problem obtained in part a).

c) What does this tell us about the form of u?

d) Use the latter expression to find a solution of (1). Check that the obtained function solves (1).

e) Alternatively, write the formula for a solution of (1) involving the heat kernel on $\mathbb{R}$. Write the heat kernel explicitly in terms of exponentials. Don't simplify the integral.

Exercise 3. *a) Show that the function $u : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}$ defined by $u(x) := \log |x|$ is harmonic on $\mathbb{R}^2 \setminus \{0\}$.*

In the following, suppose that $\phi : \mathbb{R}^2 \to \mathbb{R}$ is a smooth function which equals zero outside of some ball centered at the origin.

b) Prove that:

$$\lim_{\epsilon \to 0} \int_{\partial B(0,\epsilon)} \Big[\log |x| \cdot \frac{\partial \phi}{\partial n} - \frac{\partial}{\partial n}\Big(\log |x| \Big) \cdot \phi(x) \Big] \, dS(x) \to 2\pi\phi(0).$$

for n being the unit normal on $\partial B(0, \epsilon)$ pointing ***towards the origin****.*

c) Use the result from part b) in order to prove:

$$\phi(0) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \log |x| \cdot \Delta\phi(x) \, dx.$$

(HINT: Use Green's second identity. Be careful to isolate the singularity.)

Exercise 4. *Let us recall the representation formula for harmonic functions in three dimensions:*

For $\Omega \subseteq \mathbb{R}^3$ *a bounded domain,* u *a harmonic function on* Ω *which extends continuously up to* $\partial\Omega$*, and* $x_0 \in \Omega$*, the following formula holds:*

$$u(x_0) = \frac{1}{4\pi} \int_{\partial\Omega} \Big[-u(x) \frac{\partial}{\partial n} \Big(\frac{1}{|x - x_0|} \Big) + \frac{1}{|x - x_0|} \frac{\partial u}{\partial n} \Big] \, dS(x).$$

Here, n *denotes the outward pointing unit normal on* $\partial\Omega$*.*

In this exercise, one is allowed to use the representation formula ***without proof****.*

a) State the mean value property for harmonic functions in three-dimensions.

b) Use the representation formula in order to prove the mean value property in three dimensions.

(HINT: Consider the special case when $\Omega = B(x_0, r) \subseteq \mathbb{R}^3$ *and recall that* $\frac{\partial u}{\partial n} = \nabla u \cdot n$*. Use this fact to show that the part of the integral involving* $\frac{\partial u}{\partial n}$ *vanishes.)*

c) State the definition of the Green's function $G(x, x_0)$ *for the Laplace operator on a three-dimensional domain* Ω *with* x_0 *a point in* Ω*.*

d) Use the representation formula and properties of the Green's function to show that the harmonic function u *defined in the beginning of the problem satisfies:*

$$u(x_0) = \int_{\partial\Omega} u(x) \cdot \frac{\partial G(x, x_0)}{\partial n} \, dS(x).$$

Exercise 5. *Throughout this exercise, we assume that* $c > 0$ *is a constant.*

a) Consider the differential operator $\frac{\partial^2}{\partial t^2} - c^2 \frac{\partial^2}{\partial x^2}$*, defined on smooth functions of* $(x, t) \in \mathbb{R} \times \mathbb{R}$*.*

Show that there exist first-order differential operators T_1 *and* T_2 *such that for all smooth functions* $u : \mathbb{R}_x \times \mathbb{R}_t \to \mathbb{R}$*, the following identity holds:*

$$\Big(\frac{\partial^2}{\partial t^2} - c^2 \frac{\partial^2}{\partial x^2} \Big) u = T_1 T_2 u.$$

b) What is the physical interpretation of the operators T_1 *and* T_2*?*

c) Using the above factorization, show that the general solution to the wave equation on $\mathbb{R}_x \times \mathbb{R}_t$*:*

$$u_{tt} - c^2 u_{xx} = 0$$

is given by:

$$u(x, t) = f(x - ct) + g(x + ct)$$

for some functions $f, g : \mathbb{R} \to \mathbb{R}$*.*

d) Check that the function u *obtained in part c) solves the wave equation. How many derivatives do the functions* f *and* g *need to have in order for this calculation to be rigorous?*