LECTURE NOTES FOR 18.155 ON CONCENTRATION COMPACTNESS AND SOLITON SOLUTIONS FOR THE NLS EQUATION

VEDRAN SOHINGER

1. Introduction

In these notes, the fundamental model we will be considering is the *Nonlinear Schrödinger Equation (NLS)*:

$$\begin{cases} iu_t + \Delta u = \pm |u|^{p-1} u, \text{on}\, \mathbb{R}^d_x \times \mathbb{R}_t \\ u|_{t=0} = \Phi \in H^1(\mathbb{R}^d) \end{cases} \tag{1}$$

Here $1 < p < \infty$ is a real number. We assume that the equation is satisfied in the weak sense. We are are also assuming that [1] for $t \in \mathbb{R}$, the function $u(\cdot, t) \in H^1(\mathbb{R}^d)$, and that the mapping $t \mapsto u(t)$ is continuous as a map from $\mathbb{R}$ to $H^1(\mathbb{R}^d)$. In particular, for all t, $u(t) \in \mathcal{S}'(\mathbb{R}^d)$, so Δu makes sense, and lies in $\mathcal{S}'(\mathbb{R}^d)$. Since $u \in L^2(\mathbb{R}^d)$, it follows that $|u|^{p-1}u \in \mathcal{S}'(\mathbb{R}^d)$. The time derivative u_t is more subtle, and it makes sense if we recall that our equation is satisfied in the weak sense.

The equation (1) occurs naturally in geometric optics and Bose-Einstein condensates. A good reference for the origins of the NLS is Terence Tao's textbook [7]. Its primary importance for us will be the fact that we can use rigorous Mathematical tools from the class to study solutions of (1). We observe that there are two possible signs for the nonlinearity. The nonlinearity with the $+$-sign is called the *defocusing nonlinearity* and the one with the $-$-sign is called the *focusing nonlinearity*. In the lecture notes, we will primarily consider the *focusing nonlinearity* I will try to explain this heuristic in Exercise 5.

The equation (1) has the following conserved quantities:

$$M(u(t)) = \int_{\mathbb{R}^d} |u(x,t)|^2 dx, \text{ (Mass)} \tag{2}$$

$$E(u(t)) = \frac{1}{2} \int_{\mathbb{R}^d} |\nabla u(x,t)|^2 dx \pm \frac{1}{p+1} \int_{\mathbb{R}^d} |u(x,t)|^{p+1} dx, \text{ (Energy)} \tag{3}$$

In other words, $M(u(t)) = M(u(0))$ and $E(u(t)) = E(u(0))$. See Exercise 1.

The term $\frac{1}{2}\int_{\mathbb{R}^d} |\nabla u(x,t)|^2 dx$ is called the *kinetic energy*, whereas the term $\pm \frac{1}{p+1} \int_{\mathbb{R}^d} |u(x,t)|^{p+1} dx$ is called the *potential energy*.

Remark 1.1. *We immediately observe a fundamental difference between the defocusing and focusing nonlinearity; for the defocusing nonlinearity, the energy is always non-negative and it gives us a uniform bound on $\|u(t)\|_{H^1}$, whereas for the focusing nonlinearity, this is no longer the case. For the significance of solutions whose energy is negative, see Exercise 10.*

Heuristically, the Δ part of the equation gives us *dispersion*, i.e. it tends to make the solution more regular. To make this heuristic precise, we observe that if v solves:

$$\begin{cases} iv_t + \Delta v = 0, \text{on}\, \mathbb{R}^d_x \times \mathbb{R}_t \\ u|_{t=0} = \Psi \in L^1(\mathbb{R}^d) \end{cases} \tag{4}$$

[1]We won't go into the details of the rigorous derivation of this statement in the notes.

then, one has:

$$\|v(t)\|_{L^\infty_x} \leq C \frac{1}{|t|^{\frac{d}{2}}} \|\Psi\|_{L^1_x} \tag{5}$$

This estimate is left as Exercise 2.

On the other hand, we see from Remark 1.1 that the focusing nonlinearity can cause behavior that can lead to potential blow-up. We will be interested in solutions to the focusing NLS in which the dispersive Laplacian term and the focusing nonlinearity *balance out*. The *balanced* solution should not exhibit any decay or any blow-up. These solutions are the *solitons*.

Definition 1.2. *We say that u is a soliton solution to the focusing NLS if u is of the form:*

$$u(x,t) = Q(x)e^{it\tau}$$

for some $Q \in H^1$ and for some $\tau > 0$[2].

We can easily check that Q then has to satisfy:

$$\Delta Q + |Q|^{p-1}Q - \tau Q = 0 \tag{6}$$

We note that $Q, \Delta Q, |Q|^{p-1}Q \in \mathcal{S}'(\mathbb{R}^d)$, so we can just view this as an equality in $\mathcal{S}'(\mathbb{R}^d)$.

The main goal of these notes is to give a variational characterization of Q as a ground state, i.e. as an energy minimizer in a class of functions having fixed mass. In addition to this, we notice the interesting fact that Q gives us the optimal constant in an appropriate Gagliardo-Nirenberg Inequality. From this fact, we will note that, for certain NLS equations, the soliton solutions really give us the threshold between globally defined solutions and blow-up solutions, thus justifying the notion of an *balanced* solution introduced earlier. The variational characterization is given in Section 2. The key is to use the method of *Concentration Compactness*, which we explain in detail. The Sharp Gagliardo-Nirenberg Inequality and a global existence criterion are given in Section 3. Some tools from Littlewood-Paley Theory are introduced in Section 4. Exercises are given in Section 5. We note that some of the exercises coming from Section 3 require use of Littlew! ood-Paley Theory. Exercises from Sections 1 and 2 don't require any tools from Littlewood-Paley Theory.

2. A variational problem

Let us fix $p \in (1, 1+\frac{4}{d})$. Let $\lambda > 0$ be arbitrary. We consider the minimization of the energy functional:

$$E(u) := \frac{1}{2}\int_{\mathbb{R}^d} |\nabla u|^2 dx - \frac{1}{p+1}\int_{\mathbb{R}^d} |u|^{p+1} dx$$

subject to the constraint $\|u\|_{L^2}^2 = \lambda$. To fix notation, let us set:

$$I_\lambda := \inf\{E(u), u \in H^1, \|u\|_{L^2}^2 = \lambda\} \tag{7}$$

Let us first observe the following:

Proposition 2.1. *For $p \in (1, 1+\frac{4}{d})$, I_λ is finite. Moreover, $I_\lambda < 0$.*

We leave the proof of this Proposition as Exercise 3.

Throughout the notes, we will frequently refer to the following result, which Michael proved in the last lecture. (An alternative derivation can be found in Exercise 12)

[2]For physical reasons which won't discuss further, we only consider positive τ. The main reason is that, for positive τ, after applying an appropriate symmetry of the NLS, we can obtain traveling wave solutions which have positive speed. For an alternative justification, see Section 3.

Theorem 2.2. *(Gagliardo-Nirenberg Inequality) Let $2 \leq q \leq \infty$, and let $s > 0$ be such that:*

$$\frac{1}{q} = \frac{1}{2} - \frac{\theta s}{d}$$

for some $\theta \in [0,1)$. Then, for any $u \in H^s(\mathbb{R}^d)$, we have:

$$\|u\|_{L^q(\mathbb{R}^d)} \leq C(d,q,s)\|u\|_{L^2(\mathbb{R}^d)}^{1-\theta}\|u\|_{\dot{H}^s(\mathbb{R}^d)}^{\theta}$$

Here, the space $\dot{H}^s$ is defined by:

$$\|f\|_{\dot{H}^s} := \Big(\int_{\mathbb{R}^d} |\xi|^{2s}|\widehat{f}(\xi)|^2 d\xi\Big)^{\frac{1}{2}}$$

where $(|\nabla|^s f)\widehat{\ }(\xi) := |\xi|^s \widehat{f}(\xi)$.

Formally, we can use Lagrange multipliers [2] to find that the minimizer $f \in H^1(\mathbb{R}^d)$ has to satisfy, for some $\kappa \in \mathbb{R}$:

$$\Delta f + |f|^{p-1} f = \kappa f \tag{8}$$

The details of this calculation are the content of Exercise 4. (If there is time, we will present this in class). One can show that, for the wanted minimizer, $\kappa > 0$. The details are outlined in Exercise 5.

The main result of this section is:

Theorem 2.3. *Let $p \in (1, 1+\frac{4}{d})$, and let $\lambda > 0$. Then, for any minimizing sequence (u_n), there exists a sequence (y_n) in $\mathbb{R}^d$ such that the sequence $(u_n(\cdot + y_n))$ is relatively compact in H^1, and its limit point solves the minimization problem (7).*

The first key ingredient of the proof of Theorem 2.3 is the following useful result, whose proof is adapted from the exposition in [9].

Proposition 2.4. *(Concentration Compactness) Let (u_n) be a sequence bounded in $H^1(\mathbb{R}^d)$ with $\|u_n\|_{L^2}^2 = \lambda > 0$. Then, there exists a subsequence (u_{n_k}) satisfying one of the following three properties [3]:*

i) *(Compactness) There exists a sequence (y_k) in $\mathbb{R}^d$ with the property that for all $\epsilon > 0$, there exists $R > 0$ such that, for all k, one has:*

$$\int_{y_k + B_R(0)} |u_{n_k}|^2 dx \geq \lambda - \epsilon$$

ii) *(Vanishing) For all $R > 0$, one has:*

$$\lim_{k\to\infty} \sup_{y\in\mathbb{R}^d} \int_{y+B_R(0)} |u_{n_k}|^2 dx = 0$$

iii) *(Dichotomy) There exists $\alpha \in (0,\lambda)$ and sequences $(u_k^1), (u_k^2)$, bounded in $H^1(\mathbb{R}^d)$, such that:*
- $\|u_{n_k} - (u_k^1 + u_k^2)\|_{L^q} \to 0$ *as $k \to \infty$ whenever $2 \leq q < \frac{2d}{d-2}$ if $d \geq 3$, and $2 \leq q < \infty$ if $d = 1, 2$.*
- $\lim_{k\to\infty} \int_{\mathbb{R}^d} |u_k^1|^2 dx - \alpha = \lim_{k\to\infty} \int_{\mathbb{R}^d} |u_k^2|^2 - (\lambda - \alpha) = 0$
- $dist(supp\, u_k^1, supp\, u_k^2) \to \infty$ *as $k \to \infty$*
- $\liminf_{k\to\infty} \int_{\mathbb{R}^d} \big(|\nabla u_{n_k}|^2 - |\nabla u_k^1|^2 - |\nabla u_k^2|^2\big) dx \geq 0.$

Remark 2.5. *We view Concentration Compactness as a way of compensating for the failure of precompactness of bounded sets in infinite-dimensional Banach spaces. In essence, we are examining what is the difference between weak and strong convergence in $H^1(\mathbb{R}^d)$.*

Remark 2.6.

[3] In practice, we will be interested in the cases when these regimes are mutually exclusive.

The case of Dichotomy says that we can split the subsequence in two subsequences which are supported far apart essentially without loss of energy*. This observation will become more clear from the proof of Theorem 2.3. For details, see* (34)*.*

Remark 2.7. *The Concentration Compactness technique is due to Pierre-Louis Lions* [6]*.*

Let us now prove Proposition 2.4.

Proof. **Step 1: Definition of the *concentration functional***
Given $n \in \mathbb{N}$, and $t \geq 0$, we define the *concentration functional* $F_n(t)$ by:

$$F_n(t) := \sup_{y \in \mathbb{R}^d} \int_{y+B_t(0)} |u_n|^2 dx \tag{9}$$

One can show that, in fact:

$$F_n(t) := \max_{y \in \mathbb{R}^d} \int_{y+B_t(0)} |u_n|^2 dx \tag{10}$$

We note that $(F_n(t))$ is a sequence of non-decreasing, non-negative uniformly bounded functions on $[0, \infty)$. Furthermore, one can show that, it has a subsequence $(F_{n_k}(t))$ which converges pointwise. For this subsequence, we define for all $t \geq 0$:

$$F(t) := \lim_{k \to \infty} F_{n_k}(t) \tag{11}$$

We leave the last two statements as Exercise 6. The function F that we obtain is non-negative and non-decreasing.

Let $\alpha := \lim_{t \to \infty} F(t)$. By construction, one then has $\alpha \in [0, \lambda]$.

Step 2: Determination of the regime
We will now see that the regimes i), ii), and iii) correspond to the cases when $\alpha = \lambda$, $\alpha = 0$, and $0 < \alpha < \lambda$, respectively. We examine each case separately.

Case 1: $\alpha = \lambda$. In this case, $\lim_{t \to \infty} F(t) = \lambda$. Let us first note that we can find $R_0 > 0$ such that, for all $k \in \mathbb{N}$, one has:

$$F_{n_k}(R_0) > \frac{\lambda}{2} \tag{12}$$

By (10), we can find a sequence (y_k) in $\mathbb{R}^d$ such that:

$$F_{n_k}(R_0) = \int_{y_k + B_{R_0}(0)} |u_{n_k}|^2 dx \tag{13}$$

We will show that this sequence (y_k) satisfies the condition of Compactness.

On the other hand, we can also find $\epsilon_k \searrow 0$, $R_k \nearrow \infty$, and a strictly increasing sequence (m_k) such that, for all $k \in \mathbb{N}$:

$$F_{n_m}(R_k) > \lambda - \epsilon_k, \text{for } m \geq m_k. \tag{14}$$

Finally, let $\epsilon > 0$ be given. We find $k_0 \in \mathbb{N}$ such that $\epsilon_{k_0} < \epsilon$. In particular, it follows that, for all $k \geq k_0$:

$$F_{n_k}(R_{k_0}) > \lambda - \epsilon, \text{for } k \geq m_{k_0}. \tag{15}$$

We now fix $k \geq k_0$. By (10), we can find $\tilde{y}_k \in \mathbb{R}^d$ such that:

$$\int_{\tilde{y}_k + B_{R_{k_0}}} |u_{n_k}|^2 dx = F_{n_k}(R_{k_0}) > \lambda - \epsilon \tag{16}$$

Combining (13), (16), and the fact that $\int_{\mathbb{R}^d} |u_{n_k}|^2 dx = \lambda$, it follows that $\tilde{y}_k + B_{R_k(0)}$ and $y_k + B_{R_0}(0)$ intersect (assuming that $\epsilon < \frac{\lambda}{2}$. Hence, we can take

$$R := R_0 + 2R_{k_0} \tag{17}$$

Hence, we will then have:

$$\int_{y_k+B_R(0)} |u_{n_k}|^2 dx > \lambda - \epsilon \tag{18}$$

We then take R possibly larger for (18) to be satisfied when $k < k_0$. We can do this since $\int_{\mathbb{R}^d} |u_{n_k}|^2 dx = \lambda$, and there are finitely many $k < k_0$. It follows that Compactness indeed occurs.

Case 2: $\alpha = 0$**.** In this case, we know that F identically vanishes. Hence, given $R, \epsilon > 0$, we can find k sufficiently large such that:

$$F_{n_k}(R) \leq \epsilon$$

By construction, it follows that for such k:

$$\sup_{y \in \mathbb{R}^d} \int_{y+B_R(0)} |u_{n_k}|^2 dx \leq \epsilon.$$

So, Vanishing occurs in this case.
The third case is slightly more involved.

Case 3: $\alpha \in (0, \lambda)$**.**

Let us note that from the convergence properties of F_{n_k} and from the definition of α, it follows that there exists $R_k \to +\infty$, and $\epsilon_k \searrow 0$, such that:

$$|F_{n_k}(R_{n_k}) - \alpha| \leq \epsilon_k, |F_{n_k}(4R_{n_k}) - \alpha| \leq \epsilon_k \tag{19}$$

We now choose $\theta, \phi \in C_c^\infty(\mathbb{R}^d)$ with $0 \leq \theta, \phi \leq 1$ and:

$$\begin{cases} \theta(x) = 1, \phi(x) = 0; |x| \leq 1 \\ \theta(x) = 0, \phi(x) = 1; |x| \geq 2 \end{cases} \tag{20}$$

We define the *rescaling*: $\theta_\mu(\cdot) := \theta(\frac{\cdot}{\mu}), \phi_\mu(\cdot) := \phi(\frac{\cdot}{\mu})$. Let us recall that by assumption, we have: $M := \sup_n \|u_n\|_{H^1} < \infty$. We now observe that for all $y \in \mathbb{R}^d$:

$$\begin{aligned} \Big| \int_{\mathbb{R}^d} \big[|\nabla(\theta_R(x+y) u_{n_k}(x))|^2 - \theta_R^2(x+y) |\nabla u_{n_k}(x)|^2 \big] dx \Big| = \\ = \Big| \int_{\mathbb{R}^d} \big[|\nabla(\theta_R(x+y))|^2 |u_{n_k}(x)|^2 + 2Re\theta_R(x+y) u_{n_k}(x) \nabla(\theta(x+y)) \nabla \overline{u}_{n_k}(x) \big] dx \Big| \\ \leq C \frac{1}{R^2} \|u_{n_k}\|_{L^2}^2 + C \frac{1}{R} \|u_{n_k}\|_{L^2} \|\nabla u_{n_k}\|_{L^2} \\ \leq \frac{C}{R} \end{aligned} \tag{21}$$

An analogous argument gives us:

$$\Big| \int_{\mathbb{R}^d} \big[|\nabla(\phi_R(x+y) u_{n_k}(x))|^2 - \phi_R^2(x+y) |\nabla u_{n_k}(x)|^2 \big] dx \Big| \leq \frac{C}{R}. \tag{22}$$

We take y_k to be such that:

$$F_{n_k}(R_k) := \int_{y_k + B_{R_k}(0)} |u_{n_k}|^2 dx. \tag{23}$$

Let us now define:

$$u_k^1 := \theta_{R_k}(\cdot + y_k) u_{n_k}, u_k^2 := \phi_{4R_k}(\cdot + y_k) u_{n_k}. \tag{24}$$

We note that:

$$\int_{\mathbb{R}^d} |u_{n_k} - (u_k^1 + u_k^2)|^2 dx \leq \int_{x \in \mathbb{R}^d; |x-y_k| \in [R_k, 4R_k]} |u_{n_k}|^2 dx =^{4}$$

$$= \int_{y_k + B_{4R_k}(0)} |u_{n_k}|^2 dx - F_{n_k}(R_k) \leq F_{n_k}(4R_k) - F_{n_k}(R_k) \to 0, \text{as}\, k \to \infty. \tag{25}$$

We observe that, by Sobolev Embedding, for q as in the assumptions:

$$\|f\|_{L^q} \leq C \|f\|_{L^2}^{1+\frac{d}{q}-\frac{d}{2}} \|\nabla f\|_{L^2}^{\frac{d}{2}-\frac{d}{q}} \tag{26}$$

By using the fact that $\|u_{n_k} - (u_k^1 + u_k^2)\|_{H^1}$ is uniformly bounded, (25) and (26), it follows that:

$$\|u_{n_k} - (u_k^1 + u_k^2)\|_{L^q} \to 0, \text{as}\, k \to \infty. \tag{27}$$

By construction, we have:

$$dist(supp\, u_k^1, supp\, u_k^2) \sim R_k \to \infty, \text{as}\, k \to \infty \tag{28}$$

Let us note that:

$$\begin{cases} \int_{\mathbb{R}^d} |u_k^1|^2 dx - \alpha \geq \int_{y_k + B_{R_k}(0)} |u_{n_k}|^2 dx - \alpha = F_{n_k}(R_k) - \alpha \to 0, \text{as}\, k \to \infty, \\ \int_{\mathbb{R}^d} |u_k^1|^2 dx - \alpha \leq \int_{y_k + B_{4R_k}(0)} |u_{n_k}|^2 dx - \alpha \leq F_{n_k}(4R_k) - \alpha \to 0, \text{as}\, k \to \infty. \end{cases} \tag{29}$$

Consequently:

$$\lim_{k \to \infty} \int_{\mathbb{R}^d} |u_k^1|^2 dx = \alpha \tag{30}$$

Since $\|u_{n_k} - (u_k^1 + u_k^2)\|_{L^2} \to 0$ as $k \to \infty$, it follows that $\|u_k^1 + u_k^2\|_{L^2}^2 \to \lambda$ as $k \to \infty$. By using the support properties of u_k^1, u_k^2 (28), it follows that:

$$\lim_{k \to \infty} \int_{\mathbb{R}^d} |u_k^2|^2 dx = \lambda - \alpha \tag{31}$$

Since $|\theta|, |\phi| \leq 1$, it follows that:

$$\int_{\mathbb{R}^d} |\nabla u_{n_k}|^2 dx \geq \int_{\mathbb{R}^d} \big(\theta_{R_k}(x + y_k))^2 |\nabla u_{n_k}(x)|^2 + \phi_{4R_k}(x + y_k))^2 |\nabla u_{n_k}(x)|^2 \big) dx$$

We now use (21) and (22) to deduce that this quantity is:

$$\geq \int_{\mathbb{R}^d} \big(|\nabla u_k^1(x)|^2 + |\nabla u_k^2(x)|^2 \big) dx + O(\frac{1}{R_k})$$

Finally, we take $\liminf_{k \to \infty}$ to deduce that:

$$\liminf_{k \to \infty} \int_{\mathbb{R}^d} \big(|\nabla u_{n_k}|^2 - |\nabla u_k^1|^2 - |\nabla u_k^2|^2 \big) dx \geq 0 \tag{32}$$

¿From (27), (28), (30), (31), and (32), it follows that Dichotomy holds in this case. □

The second key ingredient to prove Theorem 2.3 is the following:

Proposition 2.8. *(Subadditivity) For all $\alpha \in (0, \lambda)$, one has:*

$$I_\lambda < I_\alpha + I_{\lambda - \alpha}.$$

[4] We note that we are using the precise choice of y_k here!

Proof. Suppose, without loss of generality that $\alpha \in (0, \frac{\lambda}{2}]$ (otherwise, we just replace α by $\lambda - \alpha$.) For $\theta \in (1, \frac{\lambda}{\alpha}]$, one has:

$$I_{\theta\alpha} = \inf_{\|u\|_{L^2}^2 = \theta\alpha} \{\frac{1}{2}\int_{\mathbb{R}^d} |\nabla u|^2 dx - \frac{1}{p+1}\int_{\mathbb{R}^d} |u|^{p+1} dx\} =$$

$$= \theta \inf_{\|u\|_{L^2}^2 = \alpha} \{\frac{1}{2}\int_{\mathbb{R}^d} |\nabla u|^2 dx - \frac{\theta^{\frac{p-1}{2}}}{p+1}\int_{\mathbb{R}^d} |u|^{p+1} dx\}$$

$$< \theta \inf_{\|u\|_{L^2}^2 = \alpha} \{\frac{1}{2}\int_{\mathbb{R}^d} |\nabla u|^2 dx - \frac{1}{p+1}\int_{\mathbb{R}^d} |u|^{p+1} dx\} = \theta I_\alpha \tag{33}$$

Hence, we can take $\theta = \frac{\lambda}{\alpha}$,and $\theta = \frac{\lambda - \alpha}{\alpha}$ and deduce that:

$$I_\lambda < \frac{\lambda}{\alpha} I_\alpha = I_\alpha + \frac{\lambda - \alpha}{\alpha} I_\alpha \leq I_\alpha + I_{\lambda - \alpha}$$

□

We now turn to the proof of Theorem 2.3.

Proof. (of Theorem 2.3)
Suppose that (u_n) is a minimizing sequence for the problem (7).

Step 1: Verify the conditions of Proposition 2.4.

By construction, $\|u_n\|_{L^2}^2 = \lambda$, for all n. One can also check that $\|\nabla u_n\|_{L^2} \leq C$ for all n. This is the content of Exercise 7.

Step 2: We show that Compactness occurs.

By Proposition 2.4, it suffices to rule out Dichotomy and Vanishing.

- Let us first observe that Dichotomy can't occur. In order to do this, we argue by contradiction. Let us suppose that (u_k^1) and (u_k^2) are as in the assumptions of Dichotomy. We can find sequences of positive real numbers (α_k) and (β_k) such that, for all k, one has:
$$\|\alpha_k u_k^1\|_{L^2}^2 = \alpha, \|\beta_k u_k^2\|_{L^2}^2 = \lambda - \alpha.$$
One then has:
$$\lim_k \alpha_k = \lim_k \beta_k = 1$$
By construction, it follows that [5]:
$$E(u_{n_k}) \geq E(u_k^1) + E(u_k^2) + \gamma_k \geq E(\alpha_k u_k^1) + E(\beta_k u_k^2) + \gamma_k' \tag{34}$$
where $\gamma_k, \gamma_k' \to 0$, as $k \to \infty$. One the other hand, we know that $E(u_{n_k}) \to I_\lambda$ as $k \to \infty$, and:
$$E(\alpha_k u_k^1) \geq I_\alpha, E(\beta_k u_k^2) \geq I_{\lambda-\alpha}.$$
We then take $\liminf_{k\to\infty}$ in (34) to obtain:
$$I_\lambda \geq I_\alpha + I_{\lambda-\alpha}.$$
This contradicts Proposition 2.8. Hence, Dichotomy can't occur.
- We now show that Vanishing can't occur. The key is to observe that Vanishing implies
$$\|u_{n_k}\|_{L^{p+1}(\mathbb{R}^d)}^{p+1} \to 0, \text{as}\, k \to \infty. \tag{35}$$
This gives a contradiction, since from (35), we could deduce that $\liminf_{k\to\infty} E(u_{n_k}) \geq 0$, and hence $I_\lambda \geq 0$, which contradicts the fact that $I_\lambda < 0$.

  We now assume that we are in the Vanishing regime, and we derive (35). Let $R > 0$ be given. Then, for all $y \in \mathbb{R}^d$, one has [6]:

[5] In this step, we see how the Dichotomy regime corresponds to splitting of the energy essentially without any loss.
[6] Here, we are using Gagliardo Nirenberg Inequality on the ball.

$$\|u_{n_k}\|^{p+1}_{L^{p+1}(y+B_R(0))} \leq C(R)\big(\|u_{n_k}\|^{(p+1)+d-\frac{d(p+1)}{2}}_{L^2(y+B_{2R}(0))}\|\nabla u_{n_k}\|^{\frac{d(p+1)}{2}-d}_{L^2(y+B_{2R}(0))}+\|u_{n_k}\|^{p+1}_{L^2(y+B_{2R}(0))}\big) \tag{36}$$

Now, we find a sequence (z_m) in $\mathbb{R}^d$ such that $(z_m+B_R(0))$ covers $\mathbb{R}^d$ and such that each point of $\mathbb{R}^d$ lies in at most $l=l(d)$ of the balls $z_m+B_{2R}(0)$. It follows that:

$$\|u_{n_k}\|^{p+1}_{L^{p+1}(\mathbb{R}^d)} \leq \sum_m \|u_{n_k}\|^{p+1}_{L^{p+1}(z_m+B_R(0))}$$

$$\leq C(R)\sum_m \big(\|u_{n_k}\|^{(p+1)+d-\frac{d(p+1)}{2}}_{L^2(z_m+B_{2R}(0))}\|\nabla u_{n_k}\|^{\frac{d(p+1)}{2}-d}_{L^2(z_m+B_{2R}(0))}+\|u_{n_k}\|^{p+1}_{L^2(z_m+B_{2R}(0))}\big)$$

Let us now take:

$$\epsilon_k := \sup_{y\in\mathbb{R}^d}\int_{y+B_{2R}(0)}|u_{n_k}|^2dx$$

Then $\epsilon_k \to 0$ as $k\to\infty$ by Vanishing. It follows that:

$$\|u_{n_k}\|^{p+1}_{L^{p+1}(\mathbb{R}^d)} \leq C(R)\epsilon_k^{\frac{p-1}{2}}\sum_m \big(\|u_{n_k}\|^{2+d-\frac{d(p+1)}{2}}_{L^2(z_m+B_{2R}(0))}\|\nabla u_{n_k}\|^{\frac{d(p+1)}{2}-d}_{L^2(z_m+B_{2R}(0))}+\|u_{n_k}\|^2_{L^2(z_m+B_{2R}(0)}\big)$$

which is:

$$\leq C(R)\epsilon_k^{\frac{p-1}{2}}\sum_m \big(\|u_{n_k}\|^2_{L^2(z_m+B_{2R}(0))}+\|\nabla u_{n_k}\|^2_{L^2(z_m+B_{2R}(0))}\big)$$

$$= C(R)l\epsilon_k^{\frac{p-1}{2}}\|u_{n_k}\|^2_{H^1(\mathbb{R}^d)} \to 0, \text{as}\, k\to\infty.$$

We note that here, we are using the fact that $\|u_{n_k}\|_{H^1(\mathbb{R}^d)}$ is uniformly bounded from Step 1.

Step 3: Conclusion of the proof. It follows from the previous arguments that Compactness occurs. In other words, there exist $y_k\in\mathbb{R}^d$ such that, for all $\epsilon>0$, there exists $R>0$ such that:

$$\int_{y_k+B_R(0)}|u_{n_k}|^2dx \geq \lambda-\epsilon \tag{37}$$

Consequently:

$$\int_{|x-y_k|>R}|u_{n_k}|^2dx \leq \epsilon \tag{38}$$

Let us consider $v_k := u_{n_k}(\cdot+y_k)$. Then, (v_k) is bounded in $H^1(\mathbb{R}^d)$, and hence by the Banach-Alaoglu Theorem, it has a weakly convergent subsequence in $H^1(\mathbb{R}^d)$, which we again label by (v_k). We recall that *Rellich's Compactness Theorem* gives us that the inclusion

$$H^1(B_R(0)) \hookrightarrow L^2(B_R(0))$$

is compact. We then use (38) with $\epsilon_n=\frac{1}{n}$, the compactness of the above inclusion, and an appropriate diagonal argument to deduce that (v_k) has a subsequence, which we again call (v_k) such that $v_k \stackrel{L^2}{\longrightarrow} v$. Since (v_k) is also weakly convergent in $H^1(\mathbb{R}^d)$, we deduce that $v_k \stackrel{H^1(\mathbb{R}^d)}{\rightharpoonup} v$. In particular, $\|v_n-v\|_{H^1(\mathbb{R}^d)}$ is uniformly bounded. We now use (26) to deduce that

$$v_k \stackrel{L^{p+1}(\mathbb{R}^d)}{\longrightarrow} v. \tag{39}$$

On the other hand, from the weak convergence in $H^1(\mathbb{R}^d)$, it follows that:

$$\int_{\mathbb{R}^d}|\nabla v|^2dx \leq \liminf_k \int_{\mathbb{R}^d}|\nabla v_k|^2dx \tag{40}$$

Combining (39) and (40), it follows that:

$$E(v) \leq \liminf_k E(v_k)$$

Consequently, v is a minimizer and $E(v)=I_\lambda$.

It remains to check that $v_k \xrightarrow{H^1} v$. ¿From the fact that $E(v_k) \to E(v)$, it follows that: $\int |\nabla v_k|^2 \to \int |\nabla v|^2 dx$. Consequently, we obtain $\|v_k\|_{H^1(\mathbb{R}^d)} \to \|v\|_{H^1(\mathbb{R}^d)}$. Since we also know that $v_k \overset{H^1(\mathbb{R}^d)}{\rightharpoonup} v$, it follows from a problem on Homework 5 that $v_k \xrightarrow{H^1(\mathbb{R}^d)} v$. □

3. The sharp constant in the Gagliardo-Nirenberg inequality and a global existence criterion

There is an alternative approach to studying the function Q [7]. We consider $p \in (1, 1 + \frac{4}{d-2})$ when $d \geq 3$ and $1 < p < \infty$ otherwise. For simplicity, let us just consider the case when $d \geq 3$. The idea, initiated by Michael Weinstein in the 1980s [8] is to study the functional:

$$W(u) := \frac{(\int_{\mathbb{R}^d} |u|^2)^{1 - \frac{(d-2)(p-1)}{4}} (\int_{\mathbb{R}^d} |\nabla u|^2)^{\frac{d(p-1)}{4}}}{\int_{\mathbb{R}^d} |u|^{p+1}} \tag{41}$$

By using the Gagliardo-Nirenberg Inequality, we observe that W is bounded from below (this is, in fact the motivation for the precise choice of W). We are interested in minimizing W, which is equivalent to finding the optimal constant in the Gagliardo-Nirenberg Inequality.

Proposition 3.1. *a) The functional W is invariant under dilations $u \mapsto \lambda_1 u$ and dilations $u \mapsto u(\lambda_2 \cdot)$, for $\lambda_1, \lambda_2 > 0$.*
b) The quantity $\frac{d}{dt}|_{t=0} W(f + t\phi)$ exists for all $f \in H^1(\mathbb{R}^d), \phi \in C_c^\infty(\mathbb{R}^d)$.
c) Suppose that $Q^* \in H^1(\mathbb{R}^d)$ is such that $\|Q^*\|_{L^2(\mathbb{R}^d)} = \|\nabla Q^*\|_{L^2(\mathbb{R}^d)} = 1$, and that Q^* is a critical point of W. Let $\alpha := \|Q^*\|^{p+1}_{L^{p+1}(\mathbb{R}^d)}$. Then, Q^* satisfies:

$$\frac{d(p-1)}{4} \Delta Q^* - (1 - \frac{(d-2)(p-1)}{4}) Q^* + \alpha \frac{p+1}{2} |Q^*|^{p-1} Q^* = 0.$$

The proof of Proposition 3.1 is left as Exercise 14.

Remark 3.2. *Let us give a vague heuristic explanation why this is not so unexpected: It might seem surprising at first that the ground state for the NLS is linked to the sharp constant in the Gagliardo-Nirenberg Inequailty! One way to explain why one would expect this type of result would be the fact that solitons (which are linked to the ground state) correspond to the situation when the kinetic and potential energy balance out. This is linked to the control of the $L^{p+1}(\mathbb{R}^d)$ norm in terms of the $\dot{H}^1$ norm of a function while keeping the $L^2(\mathbb{R}^d)$ norm fixed. The latter problem has a natural connection to the Gagliardo-Nirenberg Inequality.*

Proposition 3.3. *We can find Q^*, as before, which is a minimizer for the functional W.*

Remark 3.4. *By replacing Q^* with $|Q^*|$, we can assume WLOG that $Q^* \geq 0$. Here, we are using the fact that taking absolute values doesn't change the $L^p(\mathbb{R}^d)$ norm of a function and that it doesn't increase the $\dot{H}^1(\mathbb{R}^d)$ norm. The latter fact was shown on the previous homework.*

The proof of Proposition 3.3 is left as a combination of Exercises 15 and 16. The approach outlined in these exercises was based on Appendix B of [7].

Throughout the continuation of the discussion, we fix $p = 1 + \frac{4}{d}$. This is the endpoint which was not covered in approach we considered earlier. The corresponding nonlinearity is called *mass critical*. For the significance of this exact nonlinearity, see Exercise 17. In this case, we will be able to prove a precise coercivity result. We take Q^* to be the minimizer from Proposition 3.3, and we let $\alpha := \|Q^*\|^{p+1}_{L^{p+1}(\mathbb{R}^d)}$. By construction, one then has that $W(u) \leq \alpha$ for all $u \in H^1(\mathbb{R}^d)$. We recall the definition of Energy from (3).

It follows that:

[7] It is not immediately obvious that the two functions are indeed the same, up to symmetries. We will not mention details of any of the uniqueness results. The interested reader can consult [5, 3].

$$E(u(t)) \geq \frac{1}{2}(1 - \frac{2\alpha}{p+1})\|u(t)\|_{L^2}^{\frac{4}{d}}\|\nabla u(t)\|_{L^2}^2$$

which by conservation of Mass (2) equals:

$$\frac{1}{2}(1 - \frac{2\alpha}{p+1}\|\Phi\|_{L^2}^{\frac{4}{d}})\|\nabla u(t)\|_{L^2}^2 \tag{42}$$

Let us now take $Q := (\frac{2}{\alpha(p+1)})^{\frac{1}{p-1}}Q^*$. Then, from Proposition 3.1, Q solves the equation:

$$\Delta Q - \kappa Q + |Q|^{p-1}Q = 0$$

Hence, we can associate to Q a soliton solution $u(x,t) = Q(x)e^{it\kappa}$.

¿From (42), we can deduce that:

$$E(u(t)) \geq \frac{1}{2}\big(1 - \frac{\|\Phi\|_{L^2}^{\frac{4}{d}}}{\|Q\|_{L^2}^{\frac{4}{d}}}\big)\|\nabla u(t)\|_{L^2}^2 \tag{43}$$

Our work then implies the following [8]:

Proposition 3.5. *If $\|\Phi\|_{L^2(\mathbb{R}^d)} < \|Q\|_{L^2(\mathbb{R}^d)}$, one has that $E(u(t)) \geq 0$ for all times t. In particular, one obtains that $\|\nabla u(t)\|_{L^2(\mathbb{R}^d)}$ is uniformly bounded, and hence $\|u(t)\|_{H^1(\mathbb{R}^d)}$ is uniformly bounded.*

Remark 3.6. *¿From Proposition 3.5, one can obtain that there exists a global solution in $H^1(\mathbb{R}^d)$ if the initial data satisfies the appropriate smallness assumption $\|\Phi\|_{L^2(\mathbb{R}^d)} < \|Q\|_{L^2(\mathbb{R}^d)}$.*

Remark 3.7. *Combining the previous result and Exercise 9, it makes sense to call $\|Q\|_{L^2(\mathbb{R}^d)}$ the "critical mass".*

4. A brief excursion into Littlewood-Paley Theory

We present a quick overview of Littlewood-Paley Theory. The exposition mostly follows Appendix A of [7]. A more detailed discussion can be found in Professor Schlag's lecture notes, which are posted on the course webpage.

Definition 4.1. *(Littlewood-Paley Projection) Let $N \in 2^{\mathbb{Z}}$ be a dyadic integer, i.e. $N = 2^j$ for some $j \in \mathbb{Z}$ Let $\phi \in C_c^\infty(\mathbb{R}^d)$ be a radial function such that $\phi = 1$ for $|x| \leq 1$, and $\phi = 0$ for $|x| \geq 2$. We define the projection operator P_N on $L^2(\mathbb{R}^d)$ by:*

$$(P_N f)\widehat{\ }(\xi) = \Big(\phi(\frac{\xi}{N}) - \phi(\frac{\xi}{2N})\Big)\widehat{f}(\xi).$$

Remark 4.2. *We note that $P_N f$ has frequency support in the dyadic annulus $\{\xi \in \mathbb{R}^d; |\xi| \sim 2^j\}$.*

Remark 4.3. *By construction, we have:*

$$f = \sum_{N \in 2^{\mathbb{Z}}} P_N f. \tag{44}$$

Given $N \in 2^{\mathbb{Z}}$, we also define the quantities:

$$P_{\leq N} f := \sum_{M \in 2^{\mathbb{Z}}; M \leq N} P_M f,\ P_N f := \sum_{M \in 2^{\mathbb{Z}}; M > N} P_M f. \tag{45}$$

The Main Theorem about Littlewood-Paley Projections is the following:

[8] One might be a bit suspicious here, since we can take a ground state Q and rescale it as in Exercise 17, so the quantity $\|Q\|_{L^2}$ might not be uniquely defined. This doesn't happen though, since by Exercise 17, the L^2 norm is invariant under the natural scaling for the equation when $p = 1 + \frac{4}{d}$.

Theorem 4.4. *(Littlewood-Paley Inequality) Suppose that $1 < p < \infty$. Then, one has:*

$$\|f\|_{L^p(\mathbb{R}^d)} \sim \|\big(\sum_{N \in 2^{\mathbb{Z}}} |P_N f|^2\big)^{\frac{1}{2}}\|_{L^p(\mathbb{R}^d)}$$

We will not present the proof here. One way to prove Theorem 4.4 is to use the more general Calderon-Zygmund Theory [1]. A probabilistic approach based on Khintchine's Inequality is explained in Professor Schlag's notes.

Remark 4.5. *The case $p = 2$ follows from Plancherel's Theorem. It is quite an interesting fact that we can obtain an estimate in other L^p spaces.*

Remark 4.6. *By using properties of the Fourier Transform, we can show that:*

$$P_N f = \Phi_N * f, \textit{ where } \Phi_N(x) = \frac{1}{N^d}\Phi\big(\frac{x}{N}\big), \textit{for some fixed } \Phi \in \mathcal{S}(\mathbb{R}^d). \tag{46}$$

By using Young's inequality and the fact that $\|\Phi_N\|_{L^1}$ is independent of N, it follows that for all $1 \leq q \leq \infty$:

$$\|P_N f\|_{L^q(\mathbb{R}^d)} \leq C\|f\|_{L^q(\mathbb{R}^d)}$$

for come $C > 0$ independent of N. This is a special instance of *Bernstein's Inequality*. For the general inequality, see Exercise 11.

5. Exercises

For the Exercises, you are allowed to refer to previous results without proof.

Exercise 1. *(Conservation laws for NLS)(1 point) Formally differentiate under the integral sign to check that the mass, and energy, defined in* (2)*, and* (3) *are conserved in time. (Differentiate under the integral sign and integrate by parts fearlessly!)*

Exercise 2. *(Dispersive estimates for Linear Schrödinger equation)(1 point) Prove estimate* (5)*. (Hint: Use the Fourier Transform and Young's Inequality.) Explain why such an estimate couldn't hold if we were to consider the NLS on the d-dimensional torus $\mathbb{T}^d$. (From this fact, we can see that the NLS is more difficult to study on periodic domains. Heuristically, there is not enough space for the solution to disperse).*

Exercise 3. *(Finiteness of I_λ)(2 points)*
a) Prove Proposition 2.1 (Hint: Use Sobolev Embedding. In order to show that $I_\lambda < 0$, take u to be of the form $g(\frac{x}{\sigma})$, where g is an appropriate Gaussian.)
b) Prove that $I_\lambda = -\infty$ if $p > 1 + \frac{4}{d}$.

Exercise 4. *(Equation for Q)(1 point) Formally use Lagrange multipliers to prove* (8)*.*

Exercise 5. *(Pohozaev Identities)(2 points) Suppose that $Q \in H^1(\mathbb{R}^d)$ solves:*

$$\Delta Q + b|Q|^{p-1}Q = aQ$$

for some $a, b \in \mathbb{R}$. Show that the following identities hold:
a) $\int_{\mathbb{R}^d} |\nabla Q|^2 dx + a\int_{\mathbb{R}^d} |Q|^2 dx = b\int_{\mathbb{R}^d} |Q|^{p+1} dx$
b) $(d-2)\int_{\mathbb{R}^d} |\nabla Q|^2 dx + da\int_{\mathbb{R}^d} |Q|^2 dx = \frac{2db}{p+1}\int_{\mathbb{R}^d} |Q|^{p+1} dx$ (Hint: multiply the equation with $x \cdot \nabla \bar{Q}$).
c) Suppose that Q is the minimizer we constructed in Theorem 2.3. Show that the Lagrange multiplier κ, which appears in the equation (8) *satisfies:*

$$\kappa := \frac{(d+2) - p(d-2)}{2\lambda(\frac{d(p-1)}{4} - 1)} I_\lambda$$

and hence, deduce that $\kappa > 0$.
Observe that part a) also tells us that we don't expect to have solitons (keeping in mind our convention on the sign of τ), unless the NLS equation is focusing.

Exercise 6. *(Properties of F_n)(2 points) Prove* (10) *and show that* (F_n) *has a subsequence which converges pointwise. (Hint: Show that* (F_n) *is equicontinuous. In order to show this fact, it is good to use the Gagliardo-Nirenberg Inequality.)*

Exercise 7. *(Uniform H^1-boundedness of minimizing sequence)(1 point) Let* (u_n) *be a minimizing sequence for the problem* (7). *Show that* $\|u_n\|_{H^1(\mathbb{R}^d)}$ *is uniformly bounded.*

Exercise 8. *(Sobolev embedding on a ball)(1 point) Give a brief explanation why* (36) *holds.*

Exercise 9. *(An explicit blow-up solution)(2 points) a) Let us consider now the case* $p = 1 + \frac{4}{d}$. *Suppose* u *solves the corresponding (focusing or defocusing) NLS . We then let* v *be defined by:*

$$v(x,t) := \frac{1}{(it)^{\frac{d}{2}}} \overline{u(\frac{1}{t}, \frac{x}{t})} e^{\frac{i|x|^2}{4t}}$$

whenever $t \neq 0$. v *is called the Pseudoconformal Transform of* u. *Show that* v *is also a solution to the same NLS.*
b) Let us now restrict our attention to the focusing case. Consider $u(t,x) = Q(x)e^{it\tau}$, *and apply the Pseudoconformal Transform to obtain a solution that blows up in* H^1*-norm as* $t \to 0$. *Construct now a translate of the solution to obtain a solution whose* H^1*-norm blows up as* $t \to T_0 > 0$. *At what rate does this solution blow up? What is the* L^2 *norm of the blow up solution? (This justifies the term "critical mass" for* $\|Q\|_{L^2(\mathbb{R}^d)}$*).*

Exercise 10. *(An alternative blow-up result)(2 points) Let us consider the focusing NLS with* $p = 1 + \frac{4}{d}$, *as in the previous exercise.*
a) Show that $\frac{d}{dt}\int_{\mathbb{R}^d} |x|^2 |u(x,t)|^2 dx$ *is a positive multiple of* $E(u)$, *i.e. the energy of* u. *(This type of result is called a virial identity.)*
b) Suppose that the initial data Φ *satisfies* $E(\Phi) < 0$. *Explain why* $\int_{\mathbb{R}^d} |x|^2|u(x,t)|^2 dx$ *becomes arbitrarily close to zero in finite time.*
c) Show that the following inequality holds on $\mathbb{R}^d$*:*

$$\|f\|^2_{L^2(\mathbb{R}^d)} \leq C \||x|f\|_{L^2(\mathbb{R}^d)} \|\nabla f\|_{L^2(\mathbb{R}^d)}$$

d) Deduce that $\|u(t)\|_{H^1}$ *blows up in finite time.*
This is called Glassey's Blow-up Argument and is due to Robert Glassey [4].

Exercise 11. *(Bernstein's Inequality)(2 points) a) (1 point) Prove the following generalization of Young's Inequality: Suppose that* $1 \leq p, q, r \leq \infty$ *are such that* $\frac{1}{p} = \frac{1}{q} + \frac{1}{r} - 1$. *Show that:*

$$\|f * g\|_{L^p} \leq \|f\|_{L^q} \|g\|_{L^r}$$

(Hint: Use the standard Young Inequality and an Interpolation argument. It is useful to recall Schur's Test.)
b) (Bernstein's Inequality) (1 point for b) and c)) Suppose that $1 \leq p \leq q \leq \infty$. *Show that:*

$$\|P_N f\|_{L^q(\mathbb{R}^d)} \leq C N^{\frac{d}{p} - \frac{d}{q}} \|P_N f\|_{L^p(\mathbb{R}^d)}$$

for some constant $C > 0$ *independent of* N.
c) Deduce that, if u *is dyadically localized in frequency to a dyadic annulus, i.e.* $u = P_N u$ *for some* $N \in 2^{\mathbb{Z}}$, *then one has the following improvement of Sobolev Embedding:*

$$\|u\|_{L^\infty(\mathbb{R}^d)} \leq C \|u\|_{H^{\frac{d}{2}}(\mathbb{R}^d)},$$

for some constant $C > 0$ *independent of the frequency localization. (We note that the loss in regularity in the general Sobolev Embedding comes from the sum in* N.*)*

Exercise 12. *(Frequency proof of Gagliardo-Nirenberg Inequality)(3 points) In this exercise, we outline how one can use Littlewood-Paley projections to prove the Gagliardo-Nirenberg Inequality.*

The version of the Gagliardo-Nirenberg Inequality we want to prove is: Let $2 \leq q \leq \infty$, *and let* $\theta \in (0,1)$ *be such that* $\frac{1}{q} = \frac{1}{2} - \frac{\theta s}{d}$. *Then, for all* $u \in H^s(\mathbb{R}^d)$, *one has:*

$$\|u\|_{L^q(\mathbb{R}^d)} \leq C(d,q,s)\|u\|_{L^2(\mathbb{R}^d)}^{1-\theta}\|u\|_{\dot{H}^s(\mathbb{R}^d)}^{\theta} \tag{47}$$

a) Use homogeneity and scaling to reduce to the case $\|u\|_{L^2(\mathbb{R}^d)} = \|u\|_{\dot{H}^s(\mathbb{R}^d)} = 1$.
b) Use the Littlewood-Paley Decomposition to deduce that: $\|u\|_{L^q(\mathbb{R}^d)} \leq \sum_{N \in 2^{\mathbb{Z}}} \|P_N u\|_{L^q(\mathbb{R}^d)}$. *How can we pass from an* L^q*-norm to an* L^2*-norm in each summand?*
c) Consider the sum $N \geq 1$ *and* $N < 1$ *separately. It is useful to recall that, by the Littlewood Paley Inequality, one has:* $\|P_N u\|_{L^2(\mathbb{R}^d)} \leq C N^{-s}\|u\|_{\dot{H}^s(\mathbb{R}^d)}$.

Exercise 13. *(Algebra properties of* H^s*)(3 points) In this exercise, we study under which assumptions the product of two elements in* $H^s(\mathbb{R}^d)$ *lies in* $H^s(\mathbb{R}^d)$.
We will outline the proof of the fact that for $s > \frac{d}{2}$, $H^s(\mathbb{R}^d)$ *is an algebra, i.e. for* $f, g \in H^s(\mathbb{R}^d)$, *one obtains* $fg \in H^s(\mathbb{R}^d)$.
a) Explain why it is sufficient to prove the estimate:

$$\|fg\|_{H^s(\mathbb{R}^d)} \leq C(s,d)\big(\|f\|_{H^s(\mathbb{R}^d)}\|g\|_{L^\infty(\mathbb{R}^d)} + \|f\|_{L^\infty(\mathbb{R}^d)}\|g\|_{H^s(\mathbb{R}^d)}\big)$$

b) Decompose fg *as* $P_{\leq 1}(fg) + \sum_{N \in 2^{\mathbb{Z}}, N>1} P_N(fg)$. *Observe that it suffices to estimate the second term, i.e. the high-frequency part of* fg.
c) Fix $N \in 2^{\mathbb{Z}}, N > 1$. *Write* $P_N(fg) = P_N((P_{<\frac{N}{8}} f)g) + \sum_{M \in 2^{\mathbb{Z}}, M > \frac{N}{8}} P_N((P_M f)g)$. *Note that in the first term, we know that* g *has to also be supported on frequencies* $\sim N$. *For the second term, recall that* $\|P_M f\|_{L^2(\mathbb{R}^d)} \leq \frac{C}{M^s}\|P_M f\|_{H^s(\mathbb{R}^d)}$. *At some point, one should estimate an expression of the form* $\sum_{N \in 2^{\mathbb{Z}}} \Big(N^{2s}\big(\sum_{M \in 2^{\mathbb{Z}}, M > \frac{N}{8}} M^{-s}\|P_M f\|_{H^s}\|g\|_{L^\infty}\big)^2\Big)$. *The best way to deal with the two sums is to use the Cauchy-Schwarz Inequality in* M *and then sum an appropriate geometric series.*

Exercise 14. *(Properties of the functional* W*)(1 point) Prove Proposition 3.1.*

Exercise 15. *(Spatial localization of minimizers (Lemma B.4 in [7])(3 points) For any* $R_0 \geq 1, \eta > 0$, *there exists* $\epsilon > 0, R_1 > R_0$ *such that whenever* Q *is normalized in the following way:*

$$\int_{\mathbb{R}^d} |Q|^2 = \int_{\mathbb{R}^d} |\nabla Q|^2 = 1$$

and it is concentrated near the origin in the sense that:

$$\int_{|x| \leq R_0} |Q|^{p+1} \geq \eta$$

and is a near minimizer meaning that:

$$W(Q) \leq W_{min} + \epsilon.$$

Then, there exists $R_1 = R_1(\eta_1)$ *such that:*

$$\int_{|x| \geq R_1} |Q|^{p+1} \leq \eta.$$

Let us give a series of hints:
i) Assume that $R_1 > R_0$ *is large enough, we want to see that for large enough* R_1, *the inequality* $\int_{|x| \geq R_1} |Q|^{p+1} > \eta$ *cannot hold. Observe that we can take* R_1 *large enough, depending on* R_0 *and* ϵ *such that we can guarantee that there exists* $10R_0 < R < \frac{R_1}{10}$ *such that:*

$$\int_{\frac{R}{10} \leq |x| \leq 10R} |Q|^{p+1} + |Q|^2 + |\nabla Q|^2 \leq \epsilon$$

ii) Split Q into two pieces, Q_1, Q_2, approximately localized near $B(0,R)$ and away from $B(0,R)$. Show that:

$$\frac{1}{W(Q)} = \int_{\mathbb{R}^d} |Q|^{p+1} \leq \frac{1}{W_{min}} \sum_{j=1}^{2} \Big(\int_{\mathbb{R}^d} |Q_j|^2\Big)^{1-\frac{(d-2)(p-1)}{4}} \Big(\int_{\mathbb{R}^d} |\nabla Q_j|^2\Big)^{\frac{d(p-1)}{4}} + O(\epsilon)$$

iii) Observe that $\sum_{j=1}^2 \int_{\mathbb{R}^d} |Q_j|^2, \sum_{j=1}^2 \int_{\mathbb{R}^d} |\nabla Q_j|^2 \leq 1+O(\epsilon)$. Argue also that $c(\eta) \leq \int_{\mathbb{R}^d} |Q_j|^2, \int_{\mathbb{R}^d} |\nabla Q_j|^2 \leq 1 - c'(\eta)$, where $c(\eta), c'(\eta) > 0$.
iv) Use Hölder's inequality, and the fact that $1 - \frac{(d-2)(p-1)}{4}$ and $\frac{d(p-1)}{4}$ are positive and sum up to greater than 1 to deduce that:

$$\sum_{j=1}^{2} \Big(\int_{\mathbb{R}^d} |Q_j|^2\Big)^{1-\frac{(d-2)(p-1)}{4}} \Big(\int_{\mathbb{R}^d} |\nabla Q_j|^2\Big)^{\frac{d(p-1}{4}} \leq 1 - c''(\eta)$$

for some $c''(\eta) > 0$ if we take η to be sufficiently small. Conclude from the construction, one obtains:

$$\frac{1}{W_{min}+\epsilon} \leq \frac{1}{W(Q)} \leq \frac{1}{W_{min}}(1-c''(\eta)) + O(\epsilon),$$

which is a contradiction.

Exercise 16. *(Existence of minimizer; originally from [8]; we follow the exposition from Theorem B5 in [7]) (3 points) Prove Proposition 3.3, using Exercise 15. We give several hints:*
i) Let $W_{max} := \frac{1}{W_{min}}$. Take a minimizing sequence (Q_n) normalized in $L^2 and \dot{H}^1$. Deduce that: $0 < W_{max}^{\frac{1}{p+1}} = \limsup_{n\to\infty} \|Q_n\|_{L^{p+1}(\mathbb{R}^d)}$, which, by using a Littlewood-Paley Decomposition is $\leq \limsup_{n\to\infty} \sum_{N\in 2^{\mathbb{Z}}} \|P_N Q_n\|_{L^{p+1}(\mathbb{R}^d)}$. Recall that from the Littlewood-Paley Theory proof of the Gagliardo-Nirenberg inequality in Exercise 12 that one has:

$$\sup_{n\to\infty} \|P_N Q_n\|_{L^{p+1}(\mathbb{R}^d)} \lesssim N^{\frac{d}{2}-\frac{d}{p+1}} min(1, N^{-1}).$$

Deduce that there exists $N_0 \in 2^{\mathbb{Z}}$ such that:

$$\limsup_{n\to\infty} \|P_{N_0} Q_n\|_{L^{p+1}(\mathbb{R}^d)} > 0$$

Why can we immediately deduce that the same holds if we replace L^{p+1} by L^{∞}? Why can we, in particular, assume that $|P_{N_0}Q_n(0)| > 0$ for all n (possibly after passing to a subsequence)?
ii) Use (46) to deduce that for some $R_0 > 0$ sufficiently large and $\eta > 0$ sufficiently small, one has that for all n:

$$\int_{|x|\leq R_0} |Q_n|^{p+1} \geq \eta.$$

The key is to observe that for $R_0 > 0$ large enough, one has: $\int_{|x|\geq R_0} |\psi_{N_0}(x) Q_n(x)| \geq c_1$ and to use Hölder's inequality.
iii) Use the previous exercise to deduce that, given $\eta' > 0$ sufficiently small, there exists $R_1 = R_1(\eta') > 0$ sufficiently large such that:

$$\limsup_{n\to\infty} \int_{|x|\geq R_1} |Q_n|^{p+1} \leq \eta'.$$

iv) Finish the argument by using Rellich's Theorem and a diagonal argument as in the proof of Theorem 2.3.

Exercise 17. *(Mass criticality of NLS) (1 point) Suppose that u solves*

$$\begin{cases} iu_t + \Delta u = \pm |u|^{\frac{4}{d}} u, on \mathbb{R}^d_x \times \mathbb{R}_t \\ u|_{t=0} = \Phi \in H^1(\mathbb{R}^d) \end{cases} \tag{48}$$

For $\lambda > 0$, we let $u_\lambda(x,t) := \frac{1}{\lambda^{\frac{d}{2}}} u\left(\frac{x}{\lambda}, \frac{t}{\lambda^2}\right)$. Show that u_λ then solves:

$$\begin{cases} i(u_\lambda)_t + \Delta u_\lambda = \pm |u_\lambda|^{\frac{4}{d}} u_\lambda, on \mathbb{R}^d_x \times \mathbb{R}_t \\ (u_\lambda)|_{t=0} = \Phi_\lambda \in H^1(\mathbb{R}^d) \end{cases} \tag{49}$$

where $\Phi_\lambda(x) := \frac{1}{\lambda^{\frac{d}{2}}} \Phi\left(\frac{x}{\lambda}\right)$. Verify that $\|\Phi_\lambda\|_{L^2} = \|\Phi\|_{L^2}$. This sort of equation is called mass-critical, since the mass is left invariant under the natural scaling of the equation [9].

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[9] We observe that the scaling takes into account the fact that "one t derivative corresponds to two x derivatives", i.e. this is the parabolic scaling.