CLASS OF 1880 EXAM, SOLUTIONS

Friday, April 5, 2013.
Each problem is worth 25 points.

Exercise 1. *Suppose that* $\square ABCD$ *is a trapezoid whose sides* AB *and* CD *are parallel. Let* S *denote the intersection of the diagonals of* $\square ABCD$*. We denote by* A_1, A_2, A_3, A_4 *the areas of the triangles* $\Delta ABS, \Delta BCS, \Delta CDS, \Delta DAS$ *respectively.*

a) Prove that: $A_2 = A_4$.
b) Moreover, prove that: $A_1 \cdot A_3 = A_2^2$
c) Let A *denote the area of* $\square ABCD$*. Prove that:* $A \geq 4A_2$.
d) What can one say about $\square ABCD$ *if* $A = 4A_2$*? Prove your claim.*

Solution:

a) Since AB is parallel to CD, it follows that the distance from the point C to the line AB equals the distance from the point D to the line AB. Hence, the areas of the triangles ΔABC and ΔABD are equal. The area of the triangle ΔABD equals $A_1 + A_2$ and the area of the triangle ΔABC equals $A_1 + A_4$. Hence $A_1 + A_2 = A_1 + A_4$, from where we deduce that $A_2 = A_4$.

b) Let $\phi := \measuredangle ASB$. Then, we know that:

$$A_1 = \frac{1}{2} \cdot |AS| \cdot |BS| \cdot \sin\phi$$
$$A_2 = \frac{1}{2} \cdot |BS| \cdot |CS| \cdot \sin(180^\circ - \phi) = \frac{1}{2} \cdot |BS| \cdot |CS| \cdot \sin\phi$$
$$A_3 = \frac{1}{2} \cdot |CS| \cdot |DS| \cdot \sin\phi$$
$$A_4 = \frac{1}{2} \cdot |DS| \cdot |AS| \cdot \sin(180^\circ - \phi) = \frac{1}{2} \cdot |DS| \cdot |AS| \cdot \sin\phi.$$

It follows that:

$$A_1 \cdot A_3 = A_2 \cdot A_4 = \frac{1}{4} \cdot |AS| \cdot |BS| \cdot |CS| \cdot |DS| \cdot \sin^2\phi.$$

Since $A_2 = A_4$ by part a), it follows that:

$$A_1 \cdot A_3 = A_2^2.$$

c) We know that:

$$A = A_1 + A_2 + A_3 + A_4 = A_1 + A_3 + 2A_2.$$

By the *Arithmetic Mean - Geometric Mean Inequality*, it follows that:

$$A_1 + A_3 \geq 2\sqrt{A_1 \cdot A_3} = 2A_2.$$

Hence, it follows that:

$$A \geq 2A_2 + 2A_2 = 4A_2.$$

d) From part c), it follows that $A = 4A_2$ if and only if $A_1 + A_3 = 2\sqrt{A_1 \cdot A_3}$. We know that $A_1 + A_3 - 2\sqrt{A_1 \cdot A_3} = (\sqrt{A_1} - \sqrt{A_3})^2$, so equality holds if and only if $A_1 = A_3$. Since $A_1 \cdot A_3 = A_2^2$, it follows that $A = 4A_2$ if and only if $A_1 = A_2 = A_3 = A_4$, which holds if and only if S is the midpoint of AC and of BD. The latter is the case if and only if $\square ABCD$ is a parallelogram. $\square$

Exercise 2. *Suppose that* A *and* B *are distinct* $n \times n$ *matrices such that:*

i) $A^3 = B^3$

ii) $A^2 \cdot B = B^2 \cdot A$.

Prove that the matrix $A^2 + B^2$ is not invertible.

Solution:

Let us note that

$$(A^2 + B^2) \cdot (A - B) = A^2 \cdot A - A^2 \cdot B + B^2 \cdot A - B^2 \cdot B = (A^3 - B^3) + (A^2 \cdot B - B^2 \cdot A) = 0.$$

If $A^2 + B^2$ were invertible, we could multiply the above equality by the inverse of $A^2 + B^2$ on the left to deduce that $A - B = 0$, which is a contradiction since the matrices A and B are distinct by assumption. It follows that $A^2 + B^2$ is not invertible. □

Exercise 3. *a) Prove that for all $n \in \mathbb{N}$ the following identity holds:*

$$1^3 + 2^3 + 3^3 + \cdots n^3 = \Big(\frac{n(n+1)}{2}\Big)^2.$$

b) Suppose that $n \in \mathbb{N}$ is a positive integer and suppose that $a_1, a_2, \ldots, a_n$ are mutually distinct positive integers. Prove that:

$$\Big(\sum_{j=1}^{n} a_j^5\Big) + \Big(\sum_{j=1}^{n} a_j^7\Big) \geq 2\Big(\sum_{j=1}^{n} a_j^3\Big)^2$$

c) When does equality hold in part b)?

Solution:

a) We argue by induction on n. The base case $n = 1$ holds since both the left and right hand side are equal to 1. For the inductive step, we assume that the claim holds for $n = k$ and we want to show that it holds for $n = k + 1$. This follows from the identity:

$$\Big(\frac{(k+1)(k+2)}{2}\Big)^2 - \Big(\frac{k(k+1)}{2}\Big)^2 = (k+1)^2 \cdot \frac{1}{4}\Big((k+2)^2 - k^2\Big) = (k+1)^2 \cdot \frac{4k+4}{4} = (k+1)^3.$$

b) We prove the claim by induction on n.

Base case: $n = 1$. Let $x := a_1$. We need to show that:

$$x^5 + x^7 \geq 2(x^3)^2.$$

This bound follows from the *Arithmetic Mean - Geometric Mean Inequality*:

$$x^5 + x^7 \geq 2\sqrt{x^5 \cdot x^7} = 2x^6 = 2(x^3)^2.$$

Here, equality holds if and only if $x = 1$.

Inductive step: We suppose that the claim holds for some $n = k \in \mathbb{N}$. We want to show that it holds for $n = k + 1$.

Suppose that $a_1, \ldots, a_k, a_{k+1}$ are mutually distinct positive integers. Let us assume, without loss of generality that $x = a_{k+1}$ is the largest element of the set $\{a_1, \ldots, a_k, a_{k+1}\}$. We then obtain:

$$2\Big(\sum_{j=1}^{k+1} a_j^3\Big)^2 = 2\Big(\sum_{j=1}^{k} a_j^3 + x^3\Big)^2 = 2\Big(\sum_{j=1}^{k} a_j^3\Big)^2 + 4x^3\Big(\sum_{j=1}^{k} a_j^3\Big) + 2x^6.$$

By the inductive assumption, this quantity is:

$$\leq \Big(\sum_{j=1}^{k} a_j^5\Big) + \Big(\sum_{j=1}^{k} a_j^7\Big) + 4x^3\Big(\sum_{j=1}^{k} a_j^3\Big) + 2x^6.$$

Since by assumption, $a_1, a_2, \ldots, a_k \leq x - 1 = a_{k+1} - 1$, we obtain that this sum is:

$$\leq \Big(\sum_{j=1}^{k} a_j^5\Big) + \Big(\sum_{j=1}^{k} a_j^7\Big) + 4x^3\Big(\sum_{j=1}^{x-1} j^3\Big) + 2x^6.$$

We now use part a) to deduce that this equals:

$$\Big(\sum_{j=1}^{k} a_j^5\Big) + \Big(\sum_{j=1}^{k} a_j^7\Big) + 4x^3 \cdot \Big(\frac{(x-1)\cdot x}{2}\Big)^2 + 2x^6 =$$

$$= \Big(\sum_{j=1}^{k} a_j^5\Big) + \Big(\sum_{j=1}^{k} a_j^7\Big) + x^7 - 2x^6 + x^5 + 2x^6 =$$

$$= \Big(\sum_{j=1}^{k} a_j^5\Big) + \Big(\sum_{j=1}^{k} a_j^7\Big) + x^7 + x^5 = \Big(\sum_{j=1}^{k+1} a_j^5\Big) + \Big(\sum_{j=1}^{k+1} a_j^7\Big)$$

since $x = a_{k+1}$. The claim now follows.

c) From the proof in part b), it follows that equality holds if and only if $\{a_1, a_2, \ldots, a_n\} = \{1, 2, \ldots, n\}$. The crucial point was that we had equality in $\sum_{j=1}^{k} a_j^3 = \sum_{j=1}^{x-1} j^3$. $\square$

Exercise 4. *Let the sequence $(a_n)_{n\geq 0}$ be defined as follows:*

i) $a_0 := 0, a_1 := 1$.

ii) *Given $a_0, a_1, a_2, \ldots, a_n$, the term a_{n+1} is defined to be the smallest non-negative integer such that there don't exist $i, j \in \{0, 1, \ldots, n\}$, with $i \leq j$ such that a_i, a_j, a_{n+1} are three consecutive terms of an arithmetic sequence, i.e. $a_i + a_{n+1} = 2a_j$.*

a) Find a_2, a_3 and a_4.
b) Prove that, for $n \geq 1$, a_n equals the n-th positive integer whose expansion in base 3 doesn't contain the digit 2.
c) Find a_{100}.

Solution:

a) We note that $a_2 = 3$ (it can't equal 2 because $a_0 = 0, a_1 = 1$). Furthermore $a_3 = 4$ and $a_4 = 9$. We note that a_4 can't equal 5 since $a_1 = 1, a_2 = 3$. It can't equal 6 since $a_0 = 0, a_3 = 3$. It can't equal 7 since $a_1 = 1, a_3 = 4$. Finally, it can't equal 8 since $a_0 = 0, a_3 = 4$. If we choose $a_4 = 9$, then the condition *ii*) will be satisfied.

b) We argue by induction. Namely, we show that, for all $k \geq 1$, $a_1, \ldots, a_k$ are the first k positive integers whose expansion in basis 3 doesn't contain the digit 2.

Base case: $k = 1$. The claim holds by condition *i*).

Inductive step: Suppose that the claim holds holds for some $k \geq 1$. We want to show that it holds for $k + 1$.

We are given $a_0, a_1, \ldots, a_k$ and we want to add a_{k+1} according to the rule *ii*). Let x denote the smallest positive integer greater than a_k which doesn't contain any digits of 2 in its base three expansion. We want to argue that $a_{k+1} = x$.

Let us first show that $a_{k+1} \leq x$. This will follow if we show that for all $0 \leq i \leq j \leq k$, the numbers a_i, a_j, x are not the consecutive terms of an arithmetic sequence, i.e. it is not the case that $a_i + x = 2a_j$. Suppose that it were the case that $a_i + x = 2a_j$ for some $0 \leq i \leq j \leq k$. Then, we note that the base three expansion of $2a_j$ contains only the digits 0 and 2. On the other hand, since x is strictly bigger than a_i, it follows that there exists a digit where x has a 1 and where a_i has a 0. Let's assume that this is the m-th digit. In particular, since x and a_i only have digits 0 and 1 in base 3, it follows that there are no carries when we add them up and so the m-th digit of $x + a_i$ must equal 1. This is a contradiction. Hence, it follows that $a_{k+1} \leq x$.

We now show that $a_{k+1} \geq x$. We again argue by contradiction. Suppose that it were the case that $a_{k+1} < x$. Since $a_{k+1} > a_k$ (otherwise, we could take $i = j = k$.), it follows that we would then obtain: $a_k < a_{k+1} < x$. By construction of x and by the inductive assumption, it follows that every positive integer which is strictly between a_k and x must contain a digit 2 in its base 3 expansion.

Let y and z denote the results of replacing every digit 2 in the base 3 expansion of a_{k+1} by a 0 and by a 1 respectively. Since a_{k+1} was assumed to contain a digit 2 in its base 2 expansion, it follows that:

$$y < z < a_{k+1}$$

and

$$y + a_{k+1} = 2z.$$

Now, y, z contain no digits 2 in their base 3 expansion by definition. Hence, by the inductive assumption, we can find $0 \leq i \leq j \leq k$ such that $y = a_i, z = a_j$. Consequently:

$$a_i + a_{k+1} = 2a_j.$$

This gives us a contradiction.

Hence, it follows that $a_{k+1} \geq x$. Combining this with the fact that $a_{k+1} \leq x$, we now obtain:

$$a_{k+1} = x.$$

The claim now follows by induction.

c) From part b), we can deduce that, for $n \geq 1$, a_n equals the n-th positive integer whose base 3 expansion doesn't contain the digit 2. In particular, a_n is the result of taking the base 2 expansion of n and replacing the basis of the number system from 2 to 3, e.g. if $n = (1010)_2$ in binary, then $a_n = (1010)_3$, in base 3. In particular, we note that $100 = 64 + 32 + 4 = 2^6 + 2^5 + 2^2 = (1100100)_2$. Hence: $a_{100} = (1100100)_3 = 3^6 + 3^5 + 3^2 = 729 + 243 + 9 = 981.\square$