PICARD RANKS OF K3 SURFACES OF BHK TYPE

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Abstract. We give an explicit formula for the Picard ranks of K3 surfaces that have Berglund-Hübsch-Krawitz (BHK) Mirrors over an algebraically closed field. These K3 surfaces are those that are certain orbifold quotients of weighted Delsarte surfaces. The proof is an updated classical approach of Shioda using rational maps to relate the transcendental lattice of a Fermat hypersurface of higher degree to that of the K3 surfaces in question. The end result shows that the Picard ranks of a K3 surface of BHK-type and its BHK mirror are intrinsically intertwined. We end with an example of BHK mirror surfaces that, over certain fields, are supersingular.

1. Introduction

A classical problem is to compute the Picard rank of a given algebraic surface. Much work has been done in recent years in order to understand a generic hypersurface in a toric Fano 3-fold [4], but one may ask about the highly symmetric hypersurfaces in a weighted projective space. In this note, we give an explicit description in order to compute the Picard rank of certain symmetric K3 surfaces that are hypersurfaces in weighted projective spaces. This is a generalization of the case of Delsarte surfaces answered by Shioda in [15].

Let k be an algebraically closed field. Take F_A to be a polynomial that is a sum of $n+1$ monomials with $n+1$ variables $x_0, \dots, x_n$ over k

$$F_A := \sum_{i=0}^{n} \prod_{j=0}^{n} x_j^{a_{ij}}. \tag{1.1}$$

Set A to be the matrix $(a_{ij})_{i,j=0}^n$.

Definition 1.1 (Section 2.2 of [7]). *The polynomial F_A above is a Delsarte polynomial if the following hold:*

1. *Each entry a_{ij} of the matrix A is a nonnegative integer;*
2. *Each column of A has at least one zero;*
3. *The vector $(1, \dots, 1)^{tr}$ is an eigenvalue of A, i.e., $\sum_{j=0}^n a_{ij} =: \lambda$ for all i; and*
4. *The matrix A is a non-degenerate matrix, i.e., $\det A \neq 0$.*

If these conditions are satisfied, then F_A cuts out a well-defined hypersurface $X_A := \{F_A = 0\}$ in projective n-space $\mathbb{P}^n$. Any hypersurface X_A constructed this way is called a Delsarte hypersurface. In particular, if $n = 3$ then we say X_A is a Delsarte surface.

Note the condition (2) implies that the surface X_A does not contain a coordinate plane and condition (3) is homogeneity of the polynomial. We want to replace the homogeneity condition with quasi-homogeneity, so we generalize Definition 1.1 to get weighted-Delsarte hypersurfaces:

Definition 1.2. *A polynomial F_A as above is a weighted Delsarte polynomial if the following hold:*

1. *Each entry a_{ij} of the matrix A is a nonnegative integer;*
2. *Each column of A has at least one zero;*

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(3) *There exists positive integers* $q_0, \dots, q_n$ *so that, for all* i, $\lambda := \sum_{j=0}^{n} q_j a_{ij}$ *for some fixed* λ*; and*

(4) *The matrix* A *is a non-degenerate matrix, i.e.,* $\det A \neq 0$.

If the conditions above are satisfied, then F_A *cuts out a well-defined hypersurface* $X_A := \{F_A = 0\}$ *in weighted projective* n*-space* $W\mathbb{P}^n(q_0, \dots, q_n)$*. Any surface* X_A *constructed this way is called a weighted Delsarte hypersurface. In particular, if* $n = 3$ *we say that* X_A *is a weighted Delsarte surface.*

It is our goal to compute the Picard ranks of certain symplectic quotients of (quasi-smooth) weighted Delsarte surfaces that are K3 surfaces. In order to solve this problem, we introduce a mirror symmetry viewpoint. In 1990, the idea of a duality of moduli spaces via a mirror was brought to light by Greene and Plesser in [11]. The mirror to a weighted Fermat variety would be found by quotienting out by certain symmetries, and then taking a crepant resolution.

In 1992, Berglund and Hübsch proposed a generalization that included any weighted Delsarte Calabi-Yau hypersurface in a weighted projective space [2]. This proposal went unfortunately under-investigated until the last five years where it was generalized by Krawitz in [13]. A generalization to weighted Delsarte K3 surfaces are K3 surfaces of BHK-type, that is, K3 surfaces that have a Berglund-Hübsch-Krawitz mirror. We will now briefly outline the construction in order to introduce the objects that we will investigate. Details are in Section 2 in the case of surfaces.

Take a weighted Delsarte polynomial F_A. The polynomial F_A cuts out a quasismooth hypersurface in a weighted projective space $X_A := Z(F_A) \subseteq W\mathbb{P}^n(q_0, \dots, q_n)$ for some positive integers q_i. Assume that the weighted degree d of the polynomial F_A is the sum $\sum_i q_i$. This implies that the weighted Delsarte hypersurface X_A is Calabi-Yau.

The group $(k^*)^{n+1}$ acts on the space $W\mathbb{P}^n(q_0, \dots, q_n)$ by coordinate-wise multiplication with a subgroup k^* that acts trivially. Define the group $SL(F_A)$ to be the elements of $(k^*)^{n+1}$ that preserve the polynomial F_A and the nonvanishing holomorphic 2-form. Choose a subgroup G of $SL(F_A)$ such that it contains $k^* \cap SL(F_A)$. Setting $\bar{G}$ to be $G/(SL(F_A) \cap k^*)$, the orbifold quotient $Z_{A,G} := X_A/\bar{G}$ is a K3 orbifold.

Berglund-Hübsch-Krawitz mirror symmetry proposes that the mirror to the orbifold $Z_{A,G}$ is related to the polynomial associated to the transposed matrix A^T, $F_{A^T} := \sum_{i=0}^{n} \prod_{j=0}^{n} z_j^{a_{ji}}$. The polynomial F_{A^T} cuts out a Calabi-Yau hypersurface $X_{A^T} := Z(F_{A^T}) \subseteq W\mathbb{P}^n(r_0, \dots, r_n)$ for some positive integers r_i. The dual group introduced by Krawitz in [13] is a group G^T which satisfies the analogous conditions for X_{A^T} that the group G does for X_A. Define the quotient group $\bar{G}^T := G^T/(SL(F_{A^T}) \cap k^*)$. The *BHK mirror* Z_{A^T,G^T} to the Calabi-Yau orbifold $Z_{A,G}$ to be the quotient $X_{A^T}/\bar{G}^T$.

There has been a flurry of activity on BHK mirrors in the past five years (for when $k = \mathbb{C}$). Chiodo and Ruan in [5] prove a mirror theorem for these $(n-1)$-dimensional Calabi-Yau orbifolds on the level of Chen-Ruan Hodge cohomology:

$$H^{p,q}_{CR}(Z_{A,G}, \mathbb{C}) = H^{(n-1)-p,q}_{CR}(Z_{A^T,G^T}, \mathbb{C}).$$

This is evidence that the orbifolds $Z_{A,G}$ and Z_{A^T,G^T} form a mirror pair in dimensions 3 and greater; however it does not tell us anything in the case of surfaces. There has been recent work of Artebani, Boissière and Sarti that tries to unify the BHK mirror story with Dolgachev-Voisin mirror symmetry in the case where the hypersurface X_A is a double cover of $\mathbb{P}^2$ [1]. Their work has been extended by Comparin, Lyons, Priddis, and Suggs to prime covers of $\mathbb{P}^2$ [6]. In this corpus of work, the authors focus on proving that the Picard groups of the K3 surfaces $Z_{A,G}$ and Z_{A^T,G^T} have polarizations by so-called mirror lattices. In particular, these lattices embed into the subgroup of the Picard groups of the BHK mirrors that are invariant under the non-symplectic automorphism induced on the K3 surface due to the fact of it being a prime cover of $\mathbb{P}^2$. The fact that it is a prime cover of $\mathbb{P}^2$ requires that the polynomial be of the form

$$F_A := x_0^{a_{00}} + \sum_{i=1}^{n} \prod_{j=1}^{n} x_j^{a_{ij}}, \tag{1.2}$$

where a_{00} is a prime number.

In this note, we drop this hypothesis and the hypothesis of working over the complex numbers and investigate the Picard ranks. The key tools that we use are Shioda maps and information about the middle cohomology of Fermat varieties. We will use a Shioda map to relate each surface of BHK-type birationally to a quotient of a higher degree Fermat hypersurface in projective space by a finite group H. We then describe the H-invariant part of the transcendental lattice of the Fermat hypersurface, which gives us the rank of the transcendental lattice of the surface of BHK type, i.e., the Lefschetz number. Recall that for an algebraic surface X, the Lefschetz number $\lambda(X)$ is defined to be

$$\lambda(X) := b_2(X) - \rho(X). \tag{1.3}$$

Take BHK mirrors surfaces $Z_{A,G}$ and Z_{A^T,G^T} as above over a field of characteristic p. Take d to be a positive integer so that the matrix dA^{-1} has only integer entries. Let $\mathfrak{I}_d(p)$ be the (very computable) subset of symmetries on a degree d Fermat hypersurface X_d defined in Equation 3.5 (See Section 3.1 for details). We then describe the rank of the Picard group. In particular, we prove the following theorem:

Theorem 1.3. *The Lefschetz numbers of the BHK mirrors $Z_{A,G}$ and Z_{A^T,G^T} are:*

$$\begin{aligned} \lambda(Z_{A,G}) &= \#(\mathfrak{I}_d(p) \cap G^T) \text{ and} \\ \lambda(Z_{A^T,G^T}) &= \#(\mathfrak{I}_d(p) \cap G). \end{aligned} \tag{1.4}$$

The surprise is that the dual group G^T associated to the BHK Mirror Z_{A^T,G^T} plays a role in the computation of the Lefschetz number of the original K3 orbifold $Z_{A,G}$. We see a nice correspondence between the mirrors in this fashion. This theorem has the following corollary:

Corollary 1.4. *The Picard ranks of the BHK mirrors $Z_{A,G}$ and Z_{A^T,G^T} are:*

$$\begin{aligned} \rho(Z_{A,G}) &= 22 - \#(\mathfrak{I}_d(p) \cap G^T) \text{ and} \\ \rho(Z_{A^T,G^T}) &= 22 - \#(\mathfrak{I}_d(p) \cap G). \end{aligned} \tag{1.5}$$

An added quick corollary is a lower bound on the Picard number of a BHK mirror is by the order of dual group G^T. Also, a great benefit to this is that the Picard number of each BHK mirror surface is now computable, once one chooses over which field one works.

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2. Varieties of BHK-Type

In this section, we will introduce the varieties which have a BHK Mirror, so-called varieties of BHK type. We will start by introducing weighted Delsarte hypersurfaces that are Calabi-Yau and then move to quotienting by certain symplectic quotients. We end the section by explaining what their BHK mirrors are.

2.1. Weighted Delsarte Surfaces.

Take F_A to be a sum of $n+1$ monomials in $n+1$ variables

$$F_A := \sum_{i=0}^{n} \prod_{j=0}^{n} x_j^{a_{ij}},$$

so that the polynomial F_A is weighted Delsarte, as defined in Definition 1.2. The zero locus of the polynomial F_A cuts out a hypersurface $X_A := \{F_A = 0\}$ in the weighted-projective space $W\mathbb{P}^n(q_0, \dots, q_n)$.

When in characteristic zero, condition (4) in Definition above implies that the hypersurface X_A is quasi-smooth, i.e., the singular locus of the hypersurface X_A is exactly the singular locus of the weighted projective space intersected with the hypersurface itself, i.e.,

$$\mathrm{Sing}(X_A) = X_A \cap \mathrm{Sing}\left(W\mathbb{P}^3(q_0, q_1, q_2, q_3)\right).$$

Remark 2.1 ([9]). *Recall that there is an explicit description of the singular locus of a weighted projective space. A point $x = (x_0, x_1, x_2, x_3) \in W\mathbb{P}^3(q_0, q_1, q_2, q_3)$ is in the singular locus of the weighted projective space if and only if the quantity $gcd(q_i : i \in I(x))$ is greater than one, where the set $I(x)$ is $\{i : x_i \neq 0\}$.*

When over an arbitrary algebraically closed field, we will add an additional condition to our hypersurface:

Definition 2.2 ([8]). *We say X_A is in general position if $codim_{X_A}(X_A \cap Sing(W\mathbb{P}^3(q_0, q_1, q_2, q_3))) = 2$.*

Lemma 2.3 ([8]). *Let X_A be a quasismooth hypersurface in general position in $W\mathbb{P}^3(q_0, q_1, q_2, q_3)$, then*

$$Sing(X_A) = X_A \cap Sing(W\mathbb{P}^3(q_0, q_1, q_2, q_3)).$$

From here on, we will assume that X_A is in general position, if over a field of positive characteristic. Given a weighted Delsarte surface X_A, we can now calculate the canonical class of its (minimal) resolution $\tilde{X}_A \dashrightarrow X_A$ to be

$$\omega_{\tilde{X}_A} \cong \mathcal{O}_{\tilde{X}_A}(m - q_0 - q_1 - q_2 - q_3).$$

So, X_A is a (possibly singular) K3 surface if $\sum_{i=0}^3 q_i = m$, or, equivalently the sum $\sum_{i=0}^3 a^{ij}$ equals 1. From this point forward, we will assume that the weighted degree m is $\sum_i q_i$, though a lot of the idea presented here work for weighted Delsarte surfaces of higher degree, if one strips away the mirror viewpoint.

2.2. **Symplectic Group Actions.** In this section, we introduce certain symplectic groups that act on a weighted Delsarte surfaces. These group actions are those outlined in the Berglund-Hübsch-Krawitz mirror construction. We first start by defining what we mean by symplectic group actions over fields that are not the complex numbers.

Definition 2.4 ([10]). *Let X be a normal surface over k. Let G be a finite group of k-automorphisms of X. Denote by Y the quotient surface X/G and by $\pi : X \longrightarrow Y$ the quotient map.*

(1) *A surface X is said to be an* orbifold K3 surface *if the canonical sheaf ω_X is isomorphic to the structure sheaf $\mathcal{O}_X$, the first cohomology class $H^1(X, \mathcal{O}_X)$ of the structure sheaf vanishes, and the canonical sheaf of the minimal resolution $\sigma : \tilde{X} \to X$ is just the pullback of the canonical sheaf of X along σ, i.e., $\omega_{\tilde{X}} \cong \sigma^*(\omega_X)$.*
(2) *We say that the quotient map $\pi : X \to Y$* contains no wild codimension one ramification *if the characteristic of k does not divide the order of the inertia group of the map π at every prime divisor of X.*
(3) *The group action of G on X is called* symplectic *if every element of G fixes the nowhere vanishing 2-form on X, i.e., $g^*\omega_X = \omega_X$ for all $g \in G$.*

Lemma 2.5. *Assume that the map $\pi : X \to Y$ above contains no wild codimension one ramification. Then the canonical sheaf of Y ω_Y is isomorphic to $(\pi_*\omega_X)^G$. If additionally, the surface X is a K3, then $\omega_Y \cong \mathcal{O}_Y$.*

We now would like to give a few facts about the above objects:

Remark 2.6. *If the characteristic of the field k does not divide the order of the group G, then the map π contains no wild codimension one ramification.*

Remark 2.7. *When working over a field of positive characteristic, there exists examples of a K3 surface X and finite group G such that G is a symplectic group acting on X and the quotient X/G is not an orbifold K3 surface. (See Example 2.8 of* [10]*).*

Consider the group of automorphisms of the polynomial F_A, denoted $\mathrm{Aut}(F_A)$:

$$\mathrm{Aut}(F_A) := \big\{(\lambda_0, \lambda_1, \lambda_2, \lambda_3) \in (k^*)^4 \big| F_A(\lambda_0 x_0, \lambda_1 x_1, \lambda_2 x_2, \lambda_3 x_3) = F_A(x_0, x_1, x_2, x_3)\big\}$$

Choose the smallest positive integer d so that the matrix $B := dA^{-1}$ has only integer entries. We can describe the elements of $\mathrm{Aut}(F_A)$ easily as being generated by the elements of the torus $(k^*)^4$ generated by

$$\rho_j = (e^{2\pi i b_{0j}}, e^{2\pi i b_{1j}}, e^{2\pi i b_{2j}}, e^{2\pi i b_{3j}}),$$

where $b_{ij} := (B)_{ij}$. The group $\mathrm{Aut}(F_A)$ does not act symplectically on the hypersurface X_A, but it has a subgroup $SL(F_A)$ that does, namely:

$$SL(F_A) := \left\{ (\lambda_0, \lambda_1, \lambda_2, \lambda_3) \in \mathrm{Aut}(F_A) \middle| \prod_{i=0}^{3} \lambda_i = 1 \right\}.$$

Every element of the group $SL(F_A)$ need not act nontrivally on the hypersurface X_A. There exists a subgroup J_{F_A}, the exponential grading operator group, of the group $SL(F_A)$ that acts trivially on the hypersurface X_A. We can describe this group as:

$$J_{F_A} := \{(\lambda^{q_0}, \lambda^{q_1}, \lambda^{q_2}, \lambda^{q_3}) \in SL(F_A) | \lambda \in k^*\} = SL(F_A) \cap k^*.$$

The group J_{F_A} is generated by the element $\rho_0\rho_1\rho_2\rho_3$. Take a group G so that $J_{F_A} \subseteq G \subseteq SL(F_A)$. Then the quotient $\bar{G} := G/J_{F_A}$ is a subgroup of the (nontrivial) automorphisms of X_A that leave the nonvanishing 2-form invariant. We then take the orbifold

$$Z_{A,G} = X_A/\bar{G}.$$

Any orbifold K3 surface that can be written as $Z_{A,G}$ for an appropriate choice of A and G are defined to be K3 surfaces of BHK type, as they have a Berglund-Hübsch-Krawitz (BHK) mirror. In the next section, we describe that mirror.

2.3. **The Berglund-Hübsch-Krawitz Mirror.** In this section, we construct the BHK mirror Z_{A^T,G^T} to the orbifold K3 surface $Z_{A,G}$ defined above. Take the transposed polynomial

$$F_{A^T} = \sum_{i=0}^{3} \prod_{j=0}^{3} X_j^{a_{ji}}. \tag{2.1}$$

It is quasihomogeneous because there exist positive integers $r_i := \sum_j b_{ji}$ so that

$$F_{A^T}(\lambda^{r_0} X_0, \ldots, \lambda^{r_3} X_3) = \lambda^{m'} F_{A^T}(X_0, \ldots, X_3), \tag{2.2}$$

where $m' = \sum_i r_i$, for all $\lambda \in k^*$. Note that the polynomial F_{A^T} cuts out a well-defined (possibly singular) K3 surface $X_{A^T} := \{F_A = 0\} \subseteq W\mathbb{P}^n(r_0, \ldots, r_3)$.

Define the diagonal automorphism group, $\mathrm{Aut}(F_{A^T})$, analogously to $\mathrm{Aut}(F_A)$. The group $\mathrm{Aut}(F_{A^T})$ is generated by

$$\rho_i^T := (e^{2\pi i b_{i0}/d}, e^{2\pi i b_{i1}}, e^{2\pi i b_{i2}}, e^{2\pi i b_{i3}}).$$

Define the dual group G^T relative to G to be

$$G^T := \left\{ \prod_{i=0}^{n} (\rho_i^T)^{s_i} \middle| s_i \in \mathbb{Z}, \text{ where } \prod_{i=0}^{n} x_i^{s_i} \text{ is } G\text{-invariant} \right\} \subseteq \mathrm{Aut}(F_{A^T}). \tag{2.3}$$

Note that the dual group G^T sits between $J_{F_{A^T}}$ and $SL(F_{A^T})$ (for details, see [1]). Define the group $\tilde{G}^T := G^T/J_{F_{A^T}}$. We have a well-defined K3 orbifold $Z_{A^T,G^T} := X_{A^T}/\tilde{G}^T \subset W\mathbb{P}^n(r_0, \ldots, r_3)/\tilde{G}^T$. The K3 orbifold Z_{A^T,G^T} is the *BHK mirror* to $Z_{A,G}$.

3. Picard Ranks of Surfaces of BHK-Type

In this section, we compute the Picard numbers of the K3 surfaces of BHK type $Z_{A,G}$ described above. We will do this by showing the surfaces are birational to certain quotients of Fermat varieties, and then relating the transcendental part of the middle cohomology of the Fermat variety to the transcendental lattice of the K3 surface. We then obtain a (perhaps surprising) result where the dual group G^T related to the BHK mirror Z_{A^T,G^T} is directly related to the Picard number of the surface $Z_{A,G}$.

3.1. **Hodge Theory on Fermat Surfaces.** In this subsection, we review Shioda's treatment of Hodge theory on Fermat Surfaces as a minor digression (see [15] and [16] for more details). This computational description of Fermat surfaces will be used in the next section in a concrete manner. Let X_d be the degree d Fermat surface $\{x_0^d + x_1^d + x_2^d + x_3^d = 0\}$ in projective 3-space $\mathbb{P}^3$. Define the groups

$$M_d = \left\{(a_0, a_1, a_2, a_3) \in (\mathbb{Z}/d\mathbb{Z})^4 | a_0 + a_1 + a_2 + a_3 \equiv 0 (\text{mod } d)\right\} \tag{3.1}$$

and

$$\mathfrak{A}_d = \{(a_0, a_1, a_2, a_3) \in M_d | a_i \not\equiv 0 \ (\text{mod } d), \text{ all } i\} . \tag{3.2}$$

If the characteristic of the field k is $p > 0$, then consider the subset of $\mathfrak{A}_d$, $\mathfrak{B}_d(p)$, that is used in the study of Fermat surfaces:

$$\mathfrak{B}_d(p) := \left\{ (b_0, b_1, b_2, b_3) \in \mathfrak{A}_d | \sum_{i=0}^{3} \sum_{j=0}^{f-1} \langle \frac{ta_i p^j}{d} \rangle = 2f \text{ for all } t \text{ such that } (t, d) = 1 \right\}, \tag{3.3}$$

where f is the order of p in $(\mathbb{Z}/d\mathbb{Z})^\times$. When the field k is of characteristic zero, then we define the set $\mathfrak{B}_d(0)$ as:

$$\mathfrak{B}_d(0) := \left\{ (b_0, b_1, b_2, b_3) \in \mathfrak{A}_d | \sum_{i=0}^{3} \langle \frac{ta_i}{d} \rangle = 2 \text{ for all } t \text{ such that } (t, d) = 1 \right\}. \tag{3.4}$$

Also, we define the subset $\mathfrak{I}_d(p)$ as the complement of $\mathfrak{B}_d(p)$ in $\mathfrak{A}_d$, i.e.,

$$\mathfrak{I}_d(p) := \mathfrak{A}_d - \mathfrak{B}_d(p). \tag{3.5}$$

We can describe the cohomology of the Fermat surface X_d by using these symmetries of the variety [16]:

$$H^2(X_d, \mathbb{Q}) = \bigoplus_{\alpha \in \mathfrak{A}_d \cup \{0\}} V(\alpha), \quad \dim V(\alpha) = 1. \tag{3.6}$$

We can decompose this cohomology to be the Neron-Severi group tensored with $\mathbb{Q}$, denoted $NS_{\mathbb{Q}}(X_d)$, and the transcendental cycles tensored with $\mathbb{Q}$, denoted $T^2_{\mathbb{Q}}(X_d)$. We can describe these groups as

$$NS_{\mathbb{Q}}(X_d) = \bigoplus_{\alpha \in \mathfrak{B}_d(p) \cup \{0\}} V(\alpha) \tag{3.7}$$

and

$$T^2_{\mathbb{Q}}(X_d) = \bigoplus_{\alpha \in \mathfrak{I}_d(p)} V(\alpha). \tag{3.8}$$

3.2. **Picard Ranks of K3 Surfaces of BHK Type.** In this section, we use the Shioda map to understand the birational geometry of the K3 surfaces of BHK type. We will compute their Lefschetz numbers which are birational invariants. This in part will tell us the rank of the transcendental lattice and consequently the Picard rank of any K3 surface of the form $Z_{A,G}$ as above.

Consider a K3 hypersurface X_A defined by the polynomial F_A as above, that sits in a weighted-projective 3-space $W\mathbb{P}^3(q_0, q_1, q_2, q_3)$. Choose a group G so that $J_{F_A} \subseteq G \subseteq SL(F_A)$. Take the quotient group $\bar{G} = G/J_{F_A}$. We then have a variety of BHK type $Z_{A,G} := X_A/\bar{G}$. Let d be the smallest positive integer d so that the matrix $B := dA^{-1}$ has only integer entries. We define the *Shioda map* ϕ_B to be the rational map

$$\phi_B : \mathbb{P}^n \dashrightarrow W\mathbb{P}^3(q_0, q_1, q_2, q_3); \tag{3.9}$$

where

$$(y_0 : y_1 : y_2 : y_3) \xmapsto{\phi_B} (x_0 : x_1 : x_2 : x_3); \qquad x_j = \prod_{k=0}^{3} y_k^{b_{jk}}. \tag{3.10}$$

Note that this map is regular if and only if the matrix of exponents A is diagonal. Take the degree d Fermat hypersurface to be X_d and the defining polynomial of the Fermat hypersurface to be $F_d := x_0^d + x_1^d + x_2^d + x_3^d$. When we restrict the map ϕ_B to this Fermat hypersurface, we get the (rational) map

$$\phi_B : X_d \dashrightarrow X_A.$$

Moreover, we compose the map ϕ_B with the quotient by the group $\bar{G}$ as above, to obtain the map

$$\phi_{B,G} : X_d \dashrightarrow Z_{A,G}.$$

A natural question is to now investigate the action of an element of the diagonal automorphism group $\mathrm{Aut}(F_d)$ with respect to the Shioda map. By a linear algebra computation, one can see that we have the following commutative diagram

$$\begin{tikzcd} X_d \arrow[r, "\mu_j"] \arrow[d, dashed, "\phi_B"'] & X_d \arrow[d, dashed, "\phi_B"'] \\ X_A \arrow[r, "\rho_j"] & X_A \end{tikzcd} \tag{3.11}$$

where μ_j is the element of $\mathrm{Aut}(F_d)$ that is associated to the map that maps

$$\begin{aligned} y_j &\longmapsto e^{2\pi i/d} y_j \text{ and} \\ y_k &\longmapsto y_k \text{ for all } k \neq j. \end{aligned} \tag{3.12}$$

Note that the elements μ_j generate the group $\mathrm{Aut}(F_d)$. One obtains the (surjective) group homomorphism

$$(\phi_B)_* : \mathrm{Aut}(F_d) \longrightarrow \mathrm{Aut}(F_A); \text{ where } \mu_j \mapsto \rho_j.$$

Define the quotient groups $\overline{\mathrm{Aut}(F_d)} := \mathrm{Aut}(F_d)/J_{F_d}$ and $\overline{\mathrm{Aut}(F_A)} := \mathrm{Aut}(F_A)/J_{F_A}$ where each element of these groups act nontrivially on X_d and X_A, respectively. We have the induced map

$$\overline{(\phi_B)_*} : \overline{\mathrm{Aut}(F_d)} \longrightarrow \overline{\mathrm{Aut}(F_A)}.$$

Consider the following proposition:

Proposition 3.1 ([3], [12])**.** *The maps ϕ_B and $\phi_{B,G}$ are birational to quotient maps. In particular, the map ϕ_B is birational to the quotient map*

$$X_d \longrightarrow X_d/(\ker \overline{(\phi_B)_*}),$$

and the map $\phi_{B,G}$ is birational to the quotient map

$$X_d \longrightarrow X_d/\left(\overline{(\phi_B)_*}\right)^{-1}(\bar{G}).$$

This result helps us understand the transcendental lattice of the K3 surface tensored with $\mathbb{Q}$. Take the following specialization of a proposition of Shioda:

Proposition 3.2 (Proposition 5 of [15])**.** *For any nonsingular, complete variety X of dimension r over k, $T^n_{\mathbb{Q}}(X)$ is a birational invariant. Further, if Γ is a finite group of automorphisms of X such that the quotient $Y = X/\Gamma$ exists, then for any resolution Y' of Y, one has:*

$$T^2_{\mathbb{Q}}(Y') \cong T^2_{\mathbb{Q}}(Y) \cong T^2_{\mathbb{Q}}(X)^{\Gamma}.$$

As we know that $Z_{A,G}$ is birational to $X_d / \left(\overline{(\phi_B)_*}\right)^{-1}(\bar{G})$, we can apply the above proposition in the context of

$$\begin{aligned} X &:= X_d \\ \Gamma &:= \left(\overline{(\phi_B)_*}\right)^{-1}(\bar{G}); \text{ and} \\ Y &:= Z_{A,G}. \end{aligned} \tag{3.13}$$

The $\mathbb{Q}$-tensored transcendental lattice $T^2_{\mathbb{Q}}(X_d)$ can be decomposed as the direct sum

$$T^2_{\mathbb{Q}}(X_d) = \bigoplus_{\alpha \in \mathfrak{I}_d(p)} V(\alpha).$$

Hence, we want to know the elements of $\mathrm{Aut}(X_d)$ that are invariant under Γ. Denote the set of such elements $L(\Gamma)$. We now will describe $L(\Gamma)$.

Define an inner pairing

$$\langle , \rangle_B : \mathbb{Z}^4_d \times \mathbb{Z}^4_d \to \mathbb{Z}^4_d$$

so that $\langle \mathbf{s}, \mathbf{h} \rangle_B := \mathbf{s}^T B \mathbf{h}$. For any group $H \subseteq \mathbb{Z}^4_d$, we can define the group

$$H^{\perp_B} := \{\mathbf{s} \in \mathbb{Z}^4_d | \langle \mathbf{s}, \mathbf{h} \rangle_B \equiv 0 \text{ for all } \mathbf{h} \in H\}.$$

Here, we set H to be the group $((\phi_B)_*)^{-1}(G)$. We can see that elements of $H^{\perp_B}$ are those invariant under the group Γ. This is because if we take the inverse image $\tilde{\Gamma}$ of the group Γ under the map $\mathrm{Aut}(F_d) \to (\mathrm{Aut}(F_d)/J_{F_d})$, then we have that $H^{\perp_B} B \tilde{\Gamma} \equiv 0$. Then, we can see by a linear algebra computation that

$$L(\Gamma) = H^{\perp_B} B = G^T.$$

So now, we have an explicit description of the transcendental lattice tensored with $\mathbb{Q}$ of the orbifold $Z_{A,G}$:

$$T^2_{\mathbb{Q}}(Z_{A,G}) = \left(\bigoplus_{\alpha \in \mathfrak{I}_d(p)} V(\alpha) \right)^{\Gamma} = \bigoplus_{\alpha \in \mathfrak{I}_d(p) \cap G^T} V(\alpha).$$

Now, we remark that by [12], the BHK mirror Z_{A^T,G^T} is analogously birational to a quotient of X_d by $H^{\perp_B}/J_{F_d}$. By tracing through the arguments above and the fact that $H = (H^{\perp_B})^{\perp_{B^T}}$, we have an analogous statement about the transcendental lattice tensored with $\mathbb{Q}$ of the BHK mirror Z_{A^T,G^T}:

$$T^2_{\mathbb{Q}}(Z_{A^T,G^T}) = \bigoplus_{\alpha \in \mathfrak{I}_d(p) \cap G} V(\alpha).$$

So there is a mirror relation on the level of Lefschetz numbers for the BHK mirrors:

Theorem 3.3. *The Lefschetz numbers of the BHK mirrors $Z_{A,G}$ and Z_{A^T,G^T} are:*

$$\begin{aligned} \lambda(Z_{A,G}) &= \#(\mathfrak{I}_d(p) \cap G^T) \text{ and} \\ \lambda(Z_{A^T,G^T}) &= \#(\mathfrak{I}_d(p) \cap G). \end{aligned} \tag{3.14}$$

As the Lefschetz numbers and Picard ranks sum to 22 for any K3 surface, we then have the following corollary:

Corollary 3.4. *The Picard ranks of the BHK mirrors $Z_{A,G}$ and Z_{A^T,G^T} are:*

$$\begin{aligned}\rho(Z_{A,G}) &= 22 - \#(\mathfrak{I}_d(p) \cap G^T) \text{ and} \\ \rho(Z_{A^T,G^T}) &= 22 - \#(\mathfrak{I}_d(p) \cap G).\end{aligned} \tag{3.15}$$

4. An Example

In this section, we give an explicit example of the computation of the Picard ranks of a K3 surface of BHK type and its BHK mirror. We follow the proof above: we describe them explicitly as birational to quotients of a Fermat hypersurface in projective 3-space $\mathbb{P}^3$ and then look at the invariant part of the transcendental lattice of the Fermat hypersurface.

Consider the polynomial F_A defined to be

$$F_A := x_0^2 x_1 + x_1^2 x_2 + x_2^6 x_3 + x_3^7.$$

This polynomial cuts out a well-defined hypersurface $X_A := \{F_A = 0\}$ in the weighted projective space $W\mathbb{P}^3(2,3,1,1)$. Note that we can check that the only critical point that it has when viewed as a map $F_A : \mathbb{C}^4 \to \mathbb{C}$ is at the origin. Note that the matrix A associated to the polynomial F_A is

$$A = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 6 & 1 \\ 0 & 0 & 0 & 7 \end{pmatrix},$$

which is invertible.

We will now comment on the symmetry groups associated to the polynomial F_A. The group of automorphisms $\mathrm{Aut}(F_A)$ can be described by being generated by one element,

$$\mathrm{Aut}(F_A) = \langle(\zeta, \zeta^{-2}, \zeta^4, \zeta^{-24})\rangle,$$

where ζ is a primitive root of unity of order 168. This group does not act symplectically on the hypersurface X_A. The group that acts symplectically on X_A is the subgroup $SL(F_A)$ that is generated by one element:

$$SL(F_A) = \langle(\zeta^8, \zeta^{-16}, \zeta^{32}, \zeta^{-24})\rangle.$$

Note that this group has elements that act trivially on the hypersurface. We note that the subgroup J_{F_A}, the so-called exponential grading operator, can be described as the subgroup

$$J_{F_A} = \langle(\zeta^{48}, \zeta^{72}, \zeta^{24}, \zeta^{24})\rangle.$$

We now have a choice of choosing a group G so that it sits between the groups J_{F_A} and $SL(F_A)$, i.e.,

$$J_{F_A} \subseteq G \subseteq SL(F_A).$$

For the purposes of this example, we will choose the group G to be equal to J_{F_A}. We then have the K3 surface $Z_{A,G} = X_A/(G/J_{F_A}) = X_A$. We now compute the BHK mirror to $Z_{A,G}$. We start by looking at the transposed polynomial

$$F_{A^T} = x_0^2 + x_0 x_1^2 + x_1 x_2^6 + x_2 x_3^7.$$

This polynomial cuts out a well-defined hypersurface $X_{A^T} := \{F_{A^T} = 0\}$ in the weighted projective space $W\mathbb{P}^3(4,2,1,1)$. We can then compute the symmetry groups on the polynomial F_{A^T}:

$$\begin{aligned}\mathrm{Aut}(F_{A^T}) &= \langle(\zeta^{84}, \zeta^{-42}, \zeta^7, \zeta^{-1})\rangle; \\ SL(F_{A^T}) &= \langle(\zeta^{84}, \zeta^{42}, \zeta^{49}, \zeta^{161})\rangle; \text{ and} \\ J_{F_{A^T}} &= \langle(\zeta^{84}, \zeta^{42}, \zeta^{21}, \zeta^{21})\rangle.\end{aligned} \tag{4.1}$$

Note that when one computes the dual group G^T to G, one finds that G^T is exactly $SL(F_{A^T})$. Take the group $\bar{G}^T$ to be the quotient $G^T/J_{F_{A^T}}$. Then the BHK mirror to the K3 surface $Z_{A,G}$ is the K3 surface $Z_{A^T,G^T} := X_{A^T}/\bar{G}^T$.

Now let us rearticulate this picture in a birational setting using Shioda maps. First, we take the integer $d = 168$, the smallest positive integer so that the matrix

$$B := dA^{-1} = \begin{pmatrix} 84 & -42 & 7 & -1 \\ 0 & 84 & -14 & 2 \\ 0 & 0 & 28 & -4 \\ 0 & 0 & 0 & 24 \end{pmatrix}$$

has only integer entries. In order to have no wild codimension one ramification and have that the orbifolds $Z_{A,G}$ and Z_{A^T,G^T} are K3 orbifolds, we assume that we are working over a field of characteristic zero or p where p is not 2, 3, or 7 (so that it does not divide 168, or the order of any group by which we quotient).

We can now define the Shioda maps associated to the matrices B and B^T to be:

$$\begin{aligned} \phi_B &: \mathbb{P}^n \dashrightarrow W\mathbb{P}^3(2,3,1,1) \\ \phi_B^T &: \mathbb{P}^n \dashrightarrow W\mathbb{P}^3(4,2,1,1) \end{aligned} \tag{4.2}$$

defined by

$$\begin{aligned} (y_0 : y_1 : y_2 : y_3) &\xmapsto{\phi_B} (y_0^{84} y_1^{-42} y_2^{7} y_3^{-1} : y_1^{84} y_2^{-14} y_3^{2} : y_2^{28} y_3^{-4} : y_3^{24}) \\ (y_0 : y_1 : y_2 : y_3) &\xmapsto{\phi_{B^T}} (y_0^{84} : y_0^{-42} y_1^{84} : y_0^{7} y_1^{-14} y_2^{28} : y_0^{-1} y_1^{2} y_2^{-4} y_3^{24}). \end{aligned} \tag{4.3}$$

Take the degree $d = 168$ Fermat hypersurface X_{168} in $\mathbb{P}^3$, i.e., $X_{168} = \{F_{168} := y_0^{168} + y_1^{168} + y_2^{168} + y_3^{168} = 0\} \subset \mathbb{P}^3$. Note that if we restrict the maps ϕ_B and ϕ_{B^T} to just the hypersurface X_{168} we get the maps $X_{168} \overset{\phi_B}{\dashrightarrow} X_A$ and $X_{168} \overset{\phi_{B^T}}{\dashrightarrow} X_{A^T}$. Further, let us construct the maps $\phi_{B,G}$ and ϕ_{B^T,G^T} by composing the maps ϕ_B and ϕ_{B^T} by the quotient maps that quotient X_A and X_{A^T} by the groups (G/J_{F_A}) and $(G^T/J_{F_{A^T}})$, respectively. We then have the following diagram of rational maps:

$$\begin{tikzcd} & X_{168} \arrow[dl, dashed, "\phi_{B,G}"'] \arrow[dr, dashed, "\phi_{B^T,G^T}"] & \\ Z_{A,G} & & Z_{A^T,G^T} \end{tikzcd}$$

One can compute the following groups:

$$\begin{aligned} H &:= \langle (\zeta,\zeta,\zeta,\zeta), (\zeta^2,1,1,1), (\zeta,\zeta^2,1,1), (1,\zeta,\zeta^6,1) \rangle; \\ H^{\perp_B} &= \langle (\zeta,\zeta,\zeta,\zeta), (\zeta^7,1,1,1), (\zeta,1,\zeta^{-2},1), (1,\zeta,\zeta^{-3},1) \rangle; \text{ and} \\ J_{F_{168}} &= \langle (\zeta,\zeta,\zeta,\zeta) \rangle. \end{aligned} \tag{4.4}$$

The maps $\phi_{B,G}$ and ϕ_{B^T,G^T} are birational to quotient maps yielding the following birational equivalences:

$$\begin{aligned} Z_{A,G} &\simeq X_{168}/(H/J_{F_{168}}); \\ Z_{A^T,G^T} &\simeq X_{168}/(H^{\perp_B}/J_{F_{168}}). \end{aligned} \tag{4.5}$$

So, we recall that we know a lot about the Picard and transcendental lattices of Fermat hypersurfaces. Note that $\mathrm{Aut}(F_{168})$ is isomorphic to $(\mathbb{Z}/168\mathbb{Z})^4$. Recall that we have the sets of elements in the group $\mathrm{Aut}(F_{168})$:

$$\begin{aligned}
&M_{168} = \left\{(a_0,a_1,a_2,a_3) \in (\mathbb{Z}/168\mathbb{Z})^4 | a_0+a_1+a_2+a_3 \equiv 0 (\mathrm{mod}\ 168)\right\};\\
&\mathfrak{A}_{168} = \{(a_0,a_1,a_2,a_3) \in M_{168} | a_i \not\equiv 0 \ (\mathrm{mod}\ d), \text{ all } i\};\\
&\mathfrak{B}_{168}(p) = \left\{(b_0,b_1,b_2,b_3) \in \mathfrak{A}_{168} | \sum_{i=0}^{3}\sum_{j=0}^{f-1} \langle \frac{ta_ip^j}{d}\rangle = 2f \text{ for all } t \text{ such that } (t,d)=1\right\}; \text{ and}\\
&\mathfrak{I}_{168}(p) = \mathfrak{A}_{168} - \mathfrak{B}_{168}(p);
\end{aligned} \tag{4.6}$$

where f is the order of $\mathrm{char}k = p$ in $(\mathbb{Z}/168\mathbb{Z})^\times$ if p is positive. When the field k is of characteristic zero, then we define the set $\mathfrak{B}_{168}(0)$ as:

$$\mathfrak{B}_{168}(0) := \left\{(b_0,b_1,b_2,b_3) \in \mathfrak{A}_{168} | \sum_{i=0}^{3} \langle \frac{ta_i}{d}\rangle = 2 \text{ for all } t \text{ such that } (t,d)=1\right\}.$$

Recall that we have a description of the transcendental lattice of X_{168} tensored with $\mathbb{Q}$:

$$T^2_{\mathbb{Q}}(X_{168}) = \bigoplus_{\alpha\in\mathfrak{I}_{168}(p)} V(\alpha).$$

So, recalling Theorem 3.2 and the birational equivalences in Equation 4.5, we have:

$$\begin{aligned}
T^2_{\mathbb{Q}}(Z_{A,G}) &= \left(\bigoplus_{\alpha\in\mathfrak{I}_{168}(p)} V(\alpha)\right)^{H/J_{F_{168}}}\\
T^2_{\mathbb{Q}}(Z_{A^T,G^T}) &= \left(\bigoplus_{\alpha\in\mathfrak{I}_{168}(p)} V(\alpha)\right)^{H^{\perp_B}/J_{F_{168}}}.
\end{aligned} \tag{4.7}$$

One can see that the elements of $\mathfrak{I}_{168}(p)$ that are invariant under the action of any element of $H/J_{F_{168}}$ are exactly those also in $H^{\perp_B}B = G^T$, by the definition of $H^{\perp_B}$. One can do the analogous thing and notice that the elements of $\mathfrak{I}_{168}(p)$ that are invariant under the action of any element of $H^{\perp_B}/J_{F_{168}}$ are those also in $(H^{\perp_B})^{\perp_{B^T}}B^T = HB^T = G$. Consequently, one has that

$$\begin{aligned}
T^2_{\mathbb{Q}}(Z_{A,G}) &= \bigoplus_{\alpha\in\mathfrak{I}_{168}(p)\cap G^T} V(\alpha)\\
T^2_{\mathbb{Q}}(Z_{A^T,G^T}) &= \bigoplus_{\alpha\in\mathfrak{I}_{168}(p)\cap G} V(\alpha).
\end{aligned} \tag{4.8}$$

This means that the Lefschetz numbers $\lambda(Z_{A,G})$ and $\lambda(Z_{A^T,G^T})$ are exactly the number of elements in the sets $\mathfrak{I}_{168}(p)\cap G^T$ and $\mathfrak{I}_{168}(p)\cap G$, respectively. As both orbifolds are K3s, we then have that the Picard numbers of each are:

$$\begin{aligned}
\rho(Z_{A,G}) &= 22 - \#(\mathfrak{I}_{168}(p)\cap G^T);\\
\rho(Z_{A^T,G^T}) &= 22 - \#(\mathfrak{I}_{168}(p)\cap G).
\end{aligned} \tag{4.9}$$

We now will compute this for a few examples of p, which is just to take every element in G^T or G and then check computationally if they are in $\mathfrak{B}_{168}(p)$ or not. We now construct a table to illustrate some potential values of Picard ranks over various fields. Note an observation that was first observed by Tate [17] that if $p \equiv 1(\mathrm{mod}\ 168)$, then $\mathfrak{I}_{168}(0) = \mathfrak{I}_{168}(p)$. Otherwise, one must actually compute $\mathfrak{I}_{168}(p)$ explicitly [14]. We now provide a table of (small) primes p that do not divide 168 and the corresponding elements that are in the sets $G \cap \mathfrak{I}_{168}(p)$ and $G^T \cap \mathfrak{I}_{168}(p)$. The primes clustered into four different groups:

p	elements in $G \cap \mathfrak{I}_{168}(p)$	elements in $G^T \cap \mathfrak{I}_{168}(p)$	$\rho(Z_{A,G})$	$\rho(Z_{A^T,G^T})$
0, 11, 29, 37, 43, 53, 67, 107, 109, 113, 137, 149, 163	(48, 72, 24, 24) (96, 144, 48, 48) (144, 48, 72, 72) (24, 120, 96, 96) (72, 24, 120, 120) (120, 96, 144, 144)	(84, 126, 147, 147) (84, 42, 105, 105) (84, 126, 63, 63) (84, 42, 21, 21)	18	16
23, 71, 79 127, 151	(48, 72, 24, 24) (96, 144, 48, 48) (144, 48, 72, 72) (24, 120, 96, 96) (72, 24, 120, 120) (120, 96, 144, 144)	none	22	16
5, 13, 17, 19, 41, 59, 61, 83 89, 97, 101, 131, 139, 157	none	(84, 126, 147, 147) (84, 42, 105, 105) (84, 126, 63, 63) (84, 42, 21, 21)	18	22
31, 47, 103, 167	none	none	22	22

It is interesting to note that there exists certain values of p where either one, neither or both of the K3 surfaces are supersingular (Picard rank is 22). Also, the order of p in $(\mathbb{Z}/168\mathbb{Z})^*$ does not indicate to which cluster of values of p that a specific value of p belongs.

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