Math 114 Fall 2011
Practice Exam 2 with Solutions

Contents

1 Problems

Question 1: Evaluate the limit

$$\lim_{(x,y)\to(-1,2)} \frac{x+1}{\sqrt{(x+1)^2+(y-2)^2}}$$

(A) 0 (B) 1 (C) -1
(D) does not exist (E) $+\infty$ (F) $1/\sqrt{5}$

SOLUTION KEY: 2.1 SOLUTION: 3.1

Question 2: Use linear approximation at $(2,-1)$ to estimate the value $f(1.8,-0.9)$ of the function $f(x,y) = \sqrt{4+2x^2+4y^2}$.

(A) 4.2 (B) 4.7 (C) 3.7
(D) 3.5 (E) 4.5 (F) none of the above

SOLUTION KEY: 2.2 SOLUTION: 3.2

Question 3: The equation

$$20xy - z^4x - y^2z = 2$$

defines z as a function of x and y. Find $\dfrac{\partial z}{\partial x}$ at the point $(1, 1, 2)$.

(A) $\dfrac{1}{5}$ (B) 22 (C) -4

(D) 0 (E) $\dfrac{4}{33}$ (F) none of the above

Solution Key: 2.3 Solution: 3.3

Question 4: Evaluate the integral

$$\int\int_D \cos(x^2 + y^2 - \pi)dxdy,$$

where $D = \{(x, y)|x^2 + y^2 \leq \pi, 0 \leq y \leq x\}$.

(A) $\dfrac{\pi}{8}$ (B) $\dfrac{\pi\sqrt{\pi}}{2}$ (C) 0

(D) $\dfrac{\sqrt{\pi}}{4}$ (E) $-\dfrac{\pi}{8}$ (F) $\dfrac{1}{\sqrt{\pi}}$

Solution Key: 2.4 Solution: 3.4

Question 5: A hiker is walking on a mountain path. The surface of the mountain is described by the equation $z = 1 - 4x^2 - 3y^2$. The positive x-axis points East and the positive y-axis points North. If the hiker begins her treck at the point $\left(\frac{1}{4}, 0, \frac{3}{4}\right)$, in which direction should she walk in order to ascend most rapidly?

(A) $\frac{1}{\sqrt{2}}\,\hat{\boldsymbol{\imath}} - \frac{1}{\sqrt{2}}\hat{\boldsymbol{\jmath}}$ (B) West

(C) North (D) $-\frac{1}{2}\,\hat{\boldsymbol{\imath}} - \frac{\sqrt{3}}{2}\,\hat{\boldsymbol{\jmath}}$

(E) South-West (F) none of the above

SOLUTION KEY: 2.5 SOLUTION: 3.5

Question 6: The maximum value of $(xy)^6$ on the ellipse $\dfrac{x^2}{4} + y^2 = 1$ occurs at a point (x, y) for which y^2 is equal to

(A) $\dfrac{\sqrt{2}}{3}$ (B) $\dfrac{1}{2}$ (C) $\dfrac{2}{3}$

(D) $\dfrac{5}{11}$ (E) $\dfrac{10}{11}$ (F) 3

SOLUTION KEY: 2.6 SOLUTION: 3.6

Question 7: Evaluate

$$\int_{-1}^{0}\int_{0}^{\sqrt{x+1}} \sqrt{3y-y^3}\,dydx.$$

(A) $\frac{\sqrt{2}}{3}$ (B) $\frac{2\sqrt{2}}{3}$ (C) $\frac{\sqrt{3}}{2}$

(D) $\frac{4\sqrt{2}}{9}$ (E) $\sqrt{2}$ (F) $2\sqrt{3}$

Solution Key: 2.7 Solution: 3.7

Question 8: Classify the critical points of the function

$$f(x,y) = x^3 + y^3 + 3x^2 - 3y^2 - 8.$$

Solution Key: 2.8 Solution: 3.8

Question 9: The position of a moving particle is given by $\overrightarrow{r}(t) = \langle 2t, 1, t^2 \rangle$. Find the tangential and normal components of the acceleration vector at time $t = 1$.

SOLUTION KEY: 2.9 SOLUTION: 3.9

Question 10: Find the parametric equations for the line tangent to the curve of intersection of the surfaces $2x^2 + y^2 = 3$ and $x^2 + y^2 - z = 1$ at the point $(1, 1, 1)$.

SOLUTION KEY: 2.10 SOLUTION: 3.10

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2 Solution key

(1) (D)

(2) (C)

(3) (E)

(4) (C)

(5) (B)

(6) (B)

(7) (D)

(8) $(0,0)$ and $(-2,2)$ are saddle points, $(0,2)$ is a local minimum and $(-2,0)$ is a local maximum.

(9) $\sqrt{2}$, $\sqrt{2}$

(10) $\overrightarrow{r}(t) = \langle 1-2t, 1+4t, 1+4t \rangle$

3 Solutions

Solution of problem 1.1: If $(x, y) \to (-1, 2)$ the numerator and the denominator of the fraction both tend to zero and so we need some additional analysis before we can determine the limit. If the limit exists then we should be able to reproduce it by approaching the point $(-1, 2)$ along any curve in the (x, y)-plane. To test the existence of the limit we try approaching $(-1, 2)$ along a line. A line through the point $(-1, 2)$ with slope k is given by the equation $y - 2 = k(x + 1)$. Restricting the function $f(x, y) = (x + 1)/\sqrt{(x + 1)^2 + (y - 2)^2}$ to this line we get a function in one variable

$$f(x, 2 + k(x + 1)) = \frac{x + 1}{\sqrt{(x + 1)^2 + k^2(x + 1)^2}} = \frac{x + 1}{|x + 1|\sqrt{1 + k^2}}.$$

If the limit L of (x, y) as $(x, y) \to (-1, 2)$ exists, then for all k the limit of $\lim_{x\to-1} f(x, 2 + k(x + 1))$ will exist and will be equal to L. Suppose that x approaches -1 from the right. Then $x + 1 \geq 0$, and so $f(x, 2 + k(x + 1)) = (1 + k^2)^{-1/2}$ for such x's. Thus L will have to be equal to

$$\lim_{x\to-1^+} f(x, 2 + k(x + 1)) = (1 + k^2)^{-1/2}.$$

But for $k = 0$ this gives $L = 1$, and for $k = 1$ this gives $L = 1/\sqrt{2}$. Thus the limit of f as $(x, y) \to (-1, 2)$ can not exist. The correct answer is (D).

Solution of problem 1.2: The partial derivatives of f are

$$f_x = \frac{2x}{\sqrt{4 + 2x^2 + 4y^2}}, \qquad f_y = \frac{4y}{\sqrt{4 + 2x^2 + 4y^2}}.$$

Evaluating at $x = 2$ and $y = -1$ we get

$$f(2, -1) = 4, \qquad f_x(2, -1) = 1, \qquad f_y(2, -1) = -1,$$

and so the linear approximation of $f(x, y)$ at $(2, -1)$ is

$$L(x, y) = 4 + (x - 2) - (y + 1) = 1 + x - y.$$

Evaluating at $x = 1.8$, $y = -0.9$ we get

$$f(1.8, -0.9) \sim L(1.8, -0.9) = 1 + 1.8 + 0.9 = 3.7.$$

Hence the correct answer is (C).

Solution of problem 1.3: Differentiating the equation with respect to x we get the equation

$$20y - 4z^3 z_x x - z^4 - y^2 z_x = 0.$$

Substituting $x = 1$, $y = 1$ and $z = 2$ we get

$$20 - 32z_x - 16 - z_x = 0,$$

and so $z_x = \dfrac{4}{33}$. The correct answer is (E).

Solution of problem 1.4: Since D is constrained by the inequalities $x^2 + y^2 \leq \pi$ and $0 \leq y \leq x$, we see that the region D is bounded by the line $y = x$, the x-axis and the circle $x^2 + y^2 = \pi$. Thus D is the sector that the rays $\theta = 0$, $\theta = \frac{\pi}{4}$ cut out of the disc $r \leq \sqrt{\pi}$. In other words, we can describe D in polar coordinates as

$$D = \left\{ (r, \theta) \,\middle|\, 0 \leq \theta \leq \frac{\pi}{4}, 0 \leq r \leq \sqrt{\pi} \right\}.$$

Now we can compute:

$$\begin{aligned} \int\int_D \cos(x^2 + y^2 - \pi) dx dy &= \int_0^{\frac{\pi}{4}} \int_0^{\sqrt{\pi}} \cos(r^2 - \pi) r dr d\theta \\ &= \frac{\pi}{8} \cdot \sin(r^2 - \pi)\big|_0^{\sqrt{\pi}} = 0. \end{aligned}$$

The correct answer is (C).

Solution of problem 1.5: The direction of most rapid increase of the altitude function $z = 1 - 4x^2 - 3y^2$ is the direction of the gradient vector $\overrightarrow{\nabla} z = -8x\,\widehat{\boldsymbol{\imath}} - 6y\,\widehat{\boldsymbol{\jmath}}$. At the point $(1/4, 0, 3/4)$ we have $\overrightarrow{\nabla} z(1/4, 0) = -2\,\widehat{\boldsymbol{\imath}}$. The unit vector in the direction of $\overrightarrow{\nabla} z(1/4, 0)$ is

$$\frac{1}{|\overrightarrow{\nabla} z(1/4,0)|}\overrightarrow{\nabla} z(1/4,0) = -\widehat{\boldsymbol{\imath}}.$$

So the hiker has to walk in the direction of the vector $-\widehat{\boldsymbol{\imath}}$, which is due West. The correct answer is (B).

Solution of problem 1.6: We want to maximize $f(x, y) = (xy)^6$ subject to the constraint $g(x, y) = x^2/4 + y^2 - 1 = 0$. The maximum occurs when

$$\begin{aligned} \overrightarrow{\nabla} f &= \lambda \overrightarrow{\nabla} g, \\ g &= 0. \end{aligned}$$

We compute $\overrightarrow{\nabla} f = 6x^5y^6\widehat{\boldsymbol{\imath}} + 6x^6y^5\widehat{\boldsymbol{\jmath}}$ and $\overrightarrow{\nabla} g = (x/2)\widehat{\boldsymbol{\imath}} + 2y\widehat{\boldsymbol{\jmath}}$. So we must solve

$$\begin{aligned} 6x^5y^6 &= \frac{\lambda x}{2} \\ 6x^6y^5 &= 2\lambda y \\ \frac{x^2}{4} + y^2 &= 1 \end{aligned}$$

Solving the first two equations for λ we get $\lambda = 12x^4y^6$ and $\lambda = 3x^6y^4$, and so $12x^4y^6 = 3x^6y^4$. Thus either $x = 0$, or $y = 0$, or $xy \neq 0$ and $x^2 = 4y^2$. Substituting the latter in the third equation we get $2y^2 = 1$. If either x or y is zero, then $f = 0$ and so the maximum of f occurs when $y^2 = 1/2$. The correct answer is (B).

Solution of problem 1.7: The domain of integration consists of all (x, y) such that $-1 \leq x \leq 0$ and $0 \leq y \leq \sqrt{x+1}$. If we solve $\sqrt{x+1} = y$ for x we get $x = y^2 - 1$ and the values $x = -1$ and $x = 0$ correspond to $y = 0$ and $y = \pm 1$ respectively. However we can not have $-1 \leq y \leq 0$

since for our region we required $y \geq 0$. Thus we must have $0 \leq y \leq 1$ and $y^2 - 1 \leq x \leq 0$. In particular

$$\begin{aligned}
\int_{-1}^{0} \int_{0}^{\sqrt{x+1}} \sqrt{3y - y^3} dy dx &= \int_{0}^{1} \int_{y^2-1}^{0} \sqrt{3y - y^3} dx dy \\
&= \int_{0}^{1} -(y^2 - 1)\sqrt{3y - y^3} dy \\
&= \int_{0}^{2} \frac{\sqrt{u}}{3} du = \frac{1}{3} \cdot \frac{2}{3} \cdot u^{\frac{3}{2}} |_0^2 = \frac{4\sqrt{2}}{9}.
\end{aligned}$$

Here at the last step we used the substitution $u = 3y - y^3$.

The correct answer is (D).

Solution of problem 1.8: The partial derivatives of f are

$$\begin{aligned}
f_x &= 3x^2 + 6x \\
f_y &= 3y^2 - 6y,
\end{aligned}$$

and the second partials are

$$\begin{aligned}
f_{xx} &= 6x + 6 \\
f_{xy} &= 0 \\
f_{yy} &= 6y - 6,
\end{aligned}$$

and so $D(x, y) = f_{xx} f_{yy} - f_{xy}^2 = 36(x + 1)(y - 1)$ Solving $f_x = 0$ and $f_y = 0$ we get $x = 0, -2$ and $y = 0, 2$. Thus we have four critical points $(0, 0)$, $(0, 2)$, $(-2, 0)$ and $(-2, 2)$ and we compute

$$\begin{aligned}
D(0, 0) &= -36, & f_{xx}(0, 0) &= 6 \\
D(0, 2) &= 36, & f_{xx}(0, 2) &= 6 \\
D(-2, 0) &= 36, & f_{xx}(-2, 0) &= -6 \\
D(-2, 2) &= -36, & f_{xx}(-2, 2) &= -6.
\end{aligned}$$

By the second derivative test we conclude that $(0, 0)$ and $(-2, 2)$ are saddle points, $(0, 2)$ is a local minimum and $(-2, 0)$ is a local maximum.

Solution of problem 1.9: Since $\overrightarrow{r}(t) = \langle 2t, 1, t^2 \rangle$ we have

$$\overrightarrow{v}(t) = \langle 2, 0, 2t \rangle,$$

$$\overrightarrow{a}(t) = \rangle 0, 0, 2 \rangle.$$

In particular we compute

$$v(t) = |\overrightarrow{v}(t)| = \sqrt{4 + 0 + 4t^2} = 2\sqrt{1 + t^2},$$

and so the tangential component of the acceleration at time t is

$$a_T = v'(t) = \frac{2t}{\sqrt{1 + t^2}}.$$

For the magnitude of the acceleration at time t we have

$$a = |\overrightarrow{a}(t)| = \sqrt{0 + 0 + 4} = 2.$$

From here we can compute the normal component of the acceleration at time t:

$$a_N = \sqrt{a^2 - a_T^2} = \sqrt{4 - \frac{4t^2}{1 + t^2}} = \frac{2}{\sqrt{1 + t^2}}.$$

So at $t = 1$ we get $a_T = \sqrt{2}$ and $a_N = \sqrt{2}$.

Solution of problem 1.10: The normal vector to the surface $f(x, y, z) = 2x^2 + y^2 - 3 = 0$ at $(1.1.1)$ is

$$\overrightarrow{\nabla} f(1, 1, 1) = ((4x)\widehat{\boldsymbol{\imath}} + 2y\widehat{\boldsymbol{\jmath}})_{|(1,1,1)} = 4\widehat{\boldsymbol{\imath}} + 2\widehat{\boldsymbol{\jmath}}.$$

Similarly, the normal vector to the surface $g(x, y, z) = x^2 + y^2 - z - 1 = 0$ is

$$\overrightarrow{\nabla} g(1, 1, 1) = ((2x)\widehat{\boldsymbol{\imath}} + (2y)\widehat{\boldsymbol{\jmath}} - \widehat{\boldsymbol{k}})_{|(1,1,1)} = 2\widehat{\boldsymbol{\imath}} + 2\widehat{\boldsymbol{\jmath}} - \widehat{\boldsymbol{k}}.$$

Consequently the tangent vector to C at $(1, 1, 1)$ is

$$\overrightarrow{\nabla} f(1, 1, 1) \times \overrightarrow{\nabla} g(1, 1, 1) = \det \begin{pmatrix} \widehat{\boldsymbol{\imath}} & \widehat{\boldsymbol{\jmath}} & \widehat{\boldsymbol{k}} \\ 4 & 2 & 0 \\ 2 & 2 & -1 \end{pmatrix} = -2\widehat{\boldsymbol{\imath}} + 4\widehat{\boldsymbol{\jmath}} + 4\widehat{\boldsymbol{k}}.$$

Thus the parametric equations of the tangent line are

$$x = 1 - 2t$$
$$y = 1 + 4t$$
$$z = 1 + 4t.$$
