Solutions to Problem Set #4 (power indices)

Q 1. Suppose license plates must be six characters: the first three letters, and the last three numbers. How many different license plates are possible?

Answer There are 26 letters to choose from and 10 numbers to choose from. Since repetition is allowed, we use the multiplication principle to get

$$26 \times 26 \times 26 \times 10 \times 10 \times 10 = 17,576,000.$$

Q 2. *Mathematics and Politics*, pp. 91, problems 6 and 7.

(6) Suppose that x has five votes, y has three votes, z has three votes, and w has two votes. Assume that eight votes are needed for passage. Calculate SSI(x), SSI(y), SSI(z), and SSI(w). Show all your work.

Answer Let's do this without brute force, although brute force would certainly work. We'll figure it out for y, z, and w, since those should be pivotal less often. Then we figure it out for x by subtraction.

First, when is y pivotal? Since eight votes are needed to pass, y will be pivotal when there are five, six, or seven votes preceding it.

- five votes: Either x is to the left of y and z and w are on the right (two ways to order them); or z and w are on the left and x is on the right (two ways to order z and w). So there are four such lists.
- six votes: This is impossible.
- seven votes: x and w are both on the left (two ways to order them), and z is on the right. So there are two such lists.

We find that y is pivotal in exactly six lists. So $\text{SSI}(y) = 6/24$.

Since z has the same number of votes as y, $\text{SSI}(z) = 6/24$ as well.

Next, when is w pivotal? Only when there are six or seven votes preceding it.

- six votes: y and z are to the left of w, and x is to the right. There are two such lists.
- seven votes: This is impossible.

Since there are only two lists, $\mathrm{SSI}(w) = 2/24$.

Finally, since the SSI's must all add up to 1, we know that $\mathrm{SSI}(x) = 10/24$.

(7) Show that Luxembourg has a Shapley-Shubik index of zero.

Answer Twelve votes are needed to pass, and since Luxembourg has only one vote, it will only be pivotal if there are exactly eleven votes preceding it.

The vote totals of the other countries are 4, 4, 4, 2, and 2. Since these are all even numbers, no combination of them can add up to an odd number such as 11. Thus there is no list in which Luxembourg is pivotal, so its Shapley-Shubik index is zero.

Q 3. Consider the following simple model of the electoral college: say there are an equal number of "red" and "blue" electoral votes, and that there are 77 electoral votes from swing states. The swing states are:

- Florida (27 votes)
- Ohio (20)
- Michigan (17)
- Colorado (9)
- New Hampshire (4)

Whichever candidate gets at least 39 votes will win the Presidency. Compute the Banzhaf index of each state.

Florida has about twelve times the population of New Hampshire. Which state benefits more from the electoral college in this example?

Answer First we have to enumerate the winning coalitions. We ask, for each state, whether it is in the coalition or not. Start with the largest.

- First suppose Florida is in the coalition. Then we need 12 votes to make it win.
  - If Ohio is in the coalition, then FO is winning. It will also be winning if we add any other states to it, and there are eight ways to do this, so we get

$$FO, FOM, FOC, FON, FOMC, FOMN, FOCN, FOMCN$$

– If Ohio is not in the coalition...

* We ask whether Michigan is in the coalition. If Michigan is in the coalition, then FM is winning, and will still be winning if we add any of the two remaining states. There are four ways to do this, so we get

$$FM, FMC, FMN, FMCN$$

* If Ohio and Michigan are not in the coalition, then we ask whether Colorado is in the coalition. If it is, then FC is not a winning coalition so we need to add New Hampshire in, and we get one winning coalition that way:

$$FCN$$

- Now suppose Florida is not in the coalition. First, if Ohio is in the coalition, then we still need 19 votes, and we can get them either by adding Michigan and Colorado (and possibly New Hampshire), or Michigan and New Hampshire alone. So there are three possibilities:

$$OMC, OMCN, OMN$$

- If neither Florida nor Ohio are in the coalition, then the three remaining states only have 30 votes between them, and cannot form a winning coalition. So we are done.

We get sixteen coalitions in all. We can list them out and use the method from the book.

Coalition	F	O	M	C	N
FO	1	1	−1	−1	−1
FOM	1	1	1	−1	−1
FOC	1	1	−1	1	−1
FON	1	1	−1	−1	1
$FOMC$	1	1	1	1	−1
$FOMN$	1	1	1	−1	1
$FOCN$	1	1	−1	1	1
$FOMCN$	1	1	1	1	1
FM	1	−1	1	−1	−1
FMC	1	−1	1	1	−1
FMN	1	−1	1	−1	1
$FMCN$	1	−1	1	1	1
FCN	1	−1	−1	1	1
OMC	−1	1	1	1	−1
$OMCN$	−1	1	1	1	1
OMN	−1	1	1	−1	1

Now we have $\mathrm{TBP}(F) = 10$, $\mathrm{TBP}(O) = 6$, $\mathrm{TBP}(M) = 6$, $\mathrm{TBP}(C) = 2$, and $\mathrm{TBP}(N) = 2$. The total Banzhaf power of all voters is $10+6+6+2+2 = 26$.

The Banzhaf indices are

$$\mathrm{BI}(F) = \frac{5}{13}, \mathrm{BI}(O) = \frac{3}{13}, \mathrm{BI}(M) = \frac{3}{13}, \mathrm{BI}(C) = \frac{1}{13}, \mathrm{BI}(N) = \frac{1}{13}.$$

Since Florida has twelve times the population of New Hampshire but only five times as much power as New Hampshire, it seems like New Hampshire benefits more.

Q 4. (An easy model of the United Nations Security Council.) Suppose there are four countries on the Security Council: the United States and three others (say, Albania, Belgium, and Congo).

A resolution will pass the Security Council if the US votes for it, and if at least two of the three other countries also vote for it. List all the winning coalitions, and then compute the Banzhaf index of each country.

Answer The winning coalitions are UAB, UBC, UAC, and $UABC$.

Coalition	U	A	B	C
UAB	1	1	1	-1
UBC	1	-1	1	1
UAC	1	1	-1	1
$UABC$	1	1	1	1

We compute that $\mathrm{TBP}(U) = 4$, $\mathrm{TBP}(A) = 2$, $\mathrm{TBP}(B) = 2$, and $\mathrm{TBP}(C) = 2$. So the total Banzhaf power of all voters is $4 + 2 + 2 + 2 = 10$.

Therefore the Banzhaf indices are

$$\mathrm{BI}(U) = 0.4,\ \mathrm{BI}(A) = 0.2,\ \mathrm{BI}(B) = 0.2,\ \mathrm{BI}(C) = 0.2.$$