Solutions to Problem Set #3 (properties of voting systems)

Q 1. *Mathematics and Politics*, p. 131, problem 12, parts (a) and (c).

Say that an alternative is a *Condorcet loser* if it would be defeated by every other alternative in the kind of one-on-one contest that takes place in sequential pairwise voting with a fixed agenda. Further, say that a social choice procedure satisfies the *Condorcet loser criterion* provided that a Condorcet loser is never among the social choices. Does the Condorcet loser criterion hold for:

(a) plurality voting?

Answer No. In plurality, an unpopular candidate can be elected if two other candidates split the vote. For example, in the Bush-Nader-Gore example, where the preference lists look like

49%	48%	3%
Bush	Gore	Nader
Nader	Nader	Gore
Gore	Bush	Bush

Bush is the Condorcet loser (51% prefer Gore over Bush in a 2-way contest, and 51% prefer Nader over Bush in a 2-way contest). However, Bush wins in plurality.

(b) the Hare system?

Answer No. However, the only way the Hare system can fail to satisfy the condition is if there is an exact tie for the lowest number of first place votes. For example, suppose there are seven voters with preference lists

3	2	2
A	B	C
B	C	B
C	A	A

Then A is a Condorcet loser, since B beats A 4–3 and C beats A 4–3. However, B and C get eliminated simultaneously since there is no majority at the beginning, and A wins through this fluke.

If there are no ties, then the Hare system *does* satisfy the Condorcet loser condition. The reason is that without ties, the only way a candidate can win is if that candidate eventually beats some other candidate in the final pairwise election, and a Condorcet loser can never do this.

Q 2. *Mathematics and Politics*, pp. 129, problems 18 and 19.

(18) An interesting variant of the Hare procedure was proposed by the psychologist Clyde Coombs. It operates exactly as the Hare system does, but instead of deleting alternatives with the fewest first place votes, it deletes those with the most last place votes. (In all other ways, it operates as does the Hare procedure.)

(a) Find the social choice according to the Coombs procedure that arises from these preference lists:

c	d	c	b	e	d	c
a	a	e	d	d	e	a
e	e	d	a	a	a	e
b	c	a	e	c	b	b
d	b	b	c	b	c	d

Answer There is no majority winner of the first place votes. Among last place votes, b has the most with three; c and d have two each. So we eliminate b. Now the preference lists are

c	d	c	d	e	d	c
a	a	e	a	d	e	a
e	e	d	e	a	a	e
d	c	a	c	c	c	d

Still no candidate has a majority of first-place votes. Now c has the most last place votes, with four. So we eliminate c and get the preference lists

a	d	e	d	e	d	a
e	a	d	a	d	e	e
d	e	a	e	a	a	d

We *still* have no candidate winning a majority of first-place votes. Candidate a has the most last place votes, with three, so we eliminate a. Thus we only have d and e left.

e	d	e	d	e	d	e
d	e	d	e	d	e	d

Now candidate e wins the Coombs election.

(b) Does the Coombs system satisfy the Pareto condition?

Answer Yes. Suppose every voter prefers A over B. Then at any stage in the election, either there is a majority of first-place votes for some candidate (and we stop), or some candidate gets eliminated for having the most last place votes. The only way B can ever have any first place votes is if A gets eliminated. But B will always have more last place votes than A, so B must get eliminated before A. Thus B cannot win.

(c) Does the Coombs system satisfy the Condorcet winner criterion?

Answer No. Use the suggested counterexample (Clinton, Perot, Bush):

35%	10%	4%	10%	21%	20%
C	C	P	P	B	B
P	B	C	B	P	C
B	P	B	C	C	P

In this example, no candidate gets a majority of first-place votes. Bush has the most last-place votes, with 39%, so Bush gets eliminated. Then the election is between Clinton and Perot, and Clinton gets a majority of 55% of the vote to win.

However, Bush is the Condorcet winner. Between Bush and Perot, Bush gets 51%; between Bush and Clinton, Bush gets 51%.

(d) Does the Coombs system satisfy monotonicity?

Answer No. Use the suggested counterexample:

35%	34%	29%	2%
C	B	P	B
P	C	B	P
B	P	C	C

No candidate has a majority to begin with, so we eliminate the candidate with the most last-place votes, which is Bush with 35%. Then Clinton easily beats Perot with 69% of the vote.

However, if the voters in the last column move Clinton up (from B,P,C to B,C,P or to C,B,P), then Perot has the highest number of last-place votes, with 36%. So Perot gets eliminated first now, and between Clinton and Bush, Bush wins with 65% of the vote. So moving Clinton up on some lists has turned Clinton from a winner into a loser. Thus the system does not satisfy monotonicity.

(e) Does the Coombs system satisfy independence of irrelevant alternatives?

Answer No. Since the Coombs system elects a majority winner in a two-person election, it cannot satisfy independence of irrelevant alternatives, by the easy impossibility theorem. (We could use a specific example, as we did in the proof of the theorem, but it's not necessary here.)

(19) Suppose we have two voters and three alternatives. Find preference lists so that one of the alternatives emerges as the social choice under the Coombs procedure, but the other two emerge as tied for the win under the Hare procedure.

Answer We must have a tie in first place, which means that both candidates will win in the Hare system, since no candidate gets eliminated. In order to get the third candidate to win the Coombs procedure, both the other candidates must be eliminated by tying for last place as well. So the only way to do this is as follows:

A	B
C	C
B	A

Q 3. *Mathematics and Politics*, p. 132, problem 20.

Say that a social choice procedure satisfies the "top condition," provided that an alternative is never among the social choices unless it occurs on top of at least one individual preference list. Prove or disprove each of the following:

(a) Plurality voting satisfies the top condition.

Answer Yes. If a candidate is not on top of any list, then that candidate cannot get the most first-place votes, so the candidate must lose.

(b) The Borda count satisfies the top condition.

Answer No. The idea is that Borda count rewards candidates with lots of second-place votes. Consider the following election.

A	B	C
D	D	D
B	C	A
C	A	B

Then A, B, and C all get $0+1+3=4$ points, while D gets $2+2+2=6$ points. So D wins the Borda count, despite not being on top of any list.

(c) The Hare system satisfies the top condition.

Answer Yes. If candidate A has no first place votes, then either there is a majority winner right away (which cannot be candidate A), or we eliminate candidate A (and possibly others) since A has the fewest first place votes. Thus A must lose.

(d) Sequential pairwise voting satisfies the top condition.

Answer No. Again consider the same example as in part (b). In this example, D is the Condorcet winner, and thus wins sequential pairwise voting (for any agenda). However, D is not on top of any list.

(e) A dictatorship satisfies the top condition.

Answer Yes. If a candidate is not on top of any list, then the candidate is not on top of the dictator's list. Thus the candidate does not win.

(f) If a procedure satisfies the top condition, then it satisfies the Pareto condition.

Answer Yes. Suppose some procedure satisfies the top condition. We will prove it satisfies Pareto as well. So suppose that every voter prefers candidate A over candidate B. Then B is not on top of any list. Since the procedure satisfies the top condition, B cannot win. This is the definition of Pareto.

Q 4. Let us define a voting system called "simulated runoff." The idea is to simulate the results of a genuine runoff election between the top two vote-getters. In simulated runoff, the winner of the election is the winner of the pairwise contest between the candidates who got the highest and second-highest number of first-place votes. (This is what would happen if a real runoff election were held.)

(a) Explain why in a three-person election, instant runoff (the Hare system) and simulated runoff will always produce the same winners.

Answer For both systems, either there is a majority winner at the beginning (in which case both produce the same winner), or one candidate is eliminated (and both systems eliminate the same candidate), and then there is a majority winner in the two-person election remaining (and both systems pick this person out).

(b) Show that in the following four-person election, instant runoff and simulated runoff produce different winners.

35%	25%	20%	10%	10%
A	C	B	B	D
B	A	C	A	C
C	B	A	C	A
D	D	D	D	B

No candidate has a majority. For instant runoff, we eliminate candidate D, who only has 10% of the first-place votes. All of the votes in the last column are now identical to those in the second column, so we add those together:

35%	35%	20%	10%
A	C	B	B
B	A	C	A
C	B	A	C

Still no candidate has a majority of first-place votes. Candidate B has the fewest, with 30%, so we eliminate B and get the preference lists:

35%	35%	20%	10%
A	C	C	A
C	A	A	C

Candidate C wins instant runoff.

In simulated runoff, we have a pairwise election between A and B, since A got 35% of first-place votes and B got 30%. In this election, A beats B with 70% of the vote. So candidate A wins simulated runoff.