Solutions to Problem Set #2 (logic and voting systems)

Q 1. Negate the sentence: "For every vote Senator Specter gets, he has to spend \$10 or one hour of volunteer time."

Answer We can let x be a vote that Specter gets, and $P(x) =$ "Specter spends \$10 to get vote x" and $Q(x) =$ "Specter spends one hour of volunteer time to get vote x." Then the statement is symbolically

$$\forall x \big(P(x) \textbf{ or } Q(x)\big).$$

The negation is simply

$$\textbf{not } \forall x \big(P(x) \textbf{ or } Q(x)\big).$$
$$\exists x \textbf{ not } \big(P(x) \textbf{ or } Q(x)\big).$$
$$\exists x \big(\textbf{not } P(x) \textbf{ and not } Q(x)\big).$$

In English, the negation of the sentence is, "Senator Specter gets at least one vote for which he has to spend neither \$10 nor one hour of volunteer time." (Remember that "nor" is equivalent to "and.")

Q 2. Let $P(x, y) =$ "x gives a contribution to y," $x =$ "corporation," and $y =$ "politician."

(a) Write the statement $\exists x \forall y P(x, y)$ in English.

Answer "There is a corporation that gives a contribution to every politician."

(b) Write the statement $\forall y \exists x P(x, y)$ in English.

Answer "Every politician gets a contribution from some corporation."

(c) Which statement implies the other one?

Answer (a) implies (b). For (a), the *same* corporation gives to all different politicians. For (b), each politician might get a contribution from a *different* corporation. Or every politician might get a contribution from the same corporation. Thus if (a) is true, then (b) is also true, but (b) can easily be true without (a) being true.

(d) Write the statement "Every corporation gives a contribution to some politician" symbolically.

Answer

$$\forall x \exists y P(x, y)$$

(e) Write the statement "A politician got a contribution from a corporation" symbolically.

Answer

$$\exists x \exists y P(x, y)$$

(The formula $\exists y \exists x P(x, y)$ would also be correct; when the two quantifiers are the same, their order does not matter.)

Q 3. Consider the following axioms. If the IMF gives a country a loan, then it will impose structural adjustment. If a country has a structural adjustment program and provides welfare benefits, then the IMF did not give it a loan. If a country does not provide welfare benefits, then it experiences high poverty.

Prove that if a country does not experience high poverty, then the IMF did not give it a loan. Use either proof by contradiction or a direct proof.

Answer Write the statements as

$P =$ "The IMF gives the country a loan"
$Q =$ "The country imposes structural adjustment"
$R =$ "The country provides welfare benefits"
$S =$ "The country experiences high poverty"

Then the axioms take the form

1. If P, then Q.
2. If Q and R, then not P.
3. If not R, then S.

We want to prove, "If not S, then not P."

(Contradiction) Assume that S is false and P is true. The contrapositive of axiom (3) is "If not S, then R." Thus we know that R is true. By axiom (1), we know that Q is also true. Thus by axiom (2), since Q and R are both true, P must be false. However, P was true by assumption. So we have a contradiction, and our assumption must be false. This proves the theorem.

(Direct proof) Assume that S is false. We want to prove that P is false. The contrapositive of axiom (3) is "If not S, then R." Thus we know that R is true. We do not know anything about Q, though. However, Q is either true or false.

Case 1: Q is true. If Q is true, then since R is also true, axiom (2) implies P is false.

Case 2: Q is false. The contrapositive of axiom (1) is "If not Q, then not P." Since Q is false, P must also be false.

In either case, we have shown that P is false. So we are done.

Q 4.

- (Chapter 5, problem 3).

  For each of the five social choice procedures described in this chapter, calculate the social choice or social choices resulting from the following sequence of individual preference lists. (For sequential pairwise voting, take the agenda to be *abcde*. For the last procedure, take the fourth person to be the dictator.

c	d	c	b	e	e	c
a	a	e	d	d	d	a
e	e	d	a	a	a	e
b	c	a	e	c	c	b
d	b	b	c	b	b	d

Answer

1. Plurality: count the first place votes.

   a: 0. b: 1. c: 3. d: 2. e: 1.

   So c wins plurality.

2. Instant runoff: count the first place votes. A majority would be 4 out of 7. Since nobody has a majority, a is eliminated. Now count the first place votes without a.

   b: 1. c: 3. d: 2. e: 1.

   Still no candidate has 4 votes. So eliminate b and e together, and count the first place votes without them. Only c and d are left.

   c: 3. d: 4.

   Thus d wins instant runoff.

3. Borda count: add up the points.

$$a\colon 3+3+1+2+2+2+3=16.$$
$$b\colon 1+0+0+4+0+1+1=7.$$
$$c\colon 4+1+4+0+1+0+4=14.$$
$$d\colon 0+4+2+3+3+4+0=16.$$
$$e\colon 2+2+3+1+4+3+2=17.$$

   So e wins Borda count.

4. Sequential pairwise: consider each election.
   a vs. b: a wins, by 6–1.
   a vs. c: a wins, by 4–3.
   a vs. d: d wins, by 5–2.
   d vs. e: e wins, by 4–3.
   So e wins sequential pairwise.

5. Dictatorship: look at the fourth voter's list.
   b wins in dictatorship.

- (Chapter 5, problem 9).

  Suppose we have three voters and four alternatives and suppose the individual preference lists are as follows:

a	c	b
b	a	d
d	b	c
c	d	a

  Show that if the social choice procedure being used is sequential pairwise voting with a fixed agenda, and, if you have agenda setting power (i.e., you get to choose the order), then you can arrange for whichever alternative you want to be the social choice.

  Answer There are six possible pairwise contests: a vs. b (a wins); a vs. c (c wins); a vs. d (a wins); b vs. c (b wins); b vs. d (b wins); and c vs. d (d wins).

  To make a win: use agenda $cdba$, for example.

  To make b win: use agenda $acdb$, for example.

  To make c win: use agenda $bdac$, for example.

  To make d win: use agenda $abcd$, for example.

  (Multiple solutions are possible.)

Q 5. (Chapter 5, problem 11.)

Show that, for a fixed sequence of individual preference lists and an odd number of voters, an alternative is a Condorcet winner if and only if it emerges as the social choice in sequential pairwise voting with a fixed agenda regardless of the agenda.

Hint: The easiest way to do this is to prove two parts separately. For the first, assume some alternative is a Condorcet winner, and show that that alternative wins sequential pairing with any agenda. For the second, assume some alternative is not a Condorcet winner. Show that there is at least one agenda that will make that person lose a sequential pairwise vote. (Construct one.)

Answer First assume some alternative A is a Condorcet winner. Then in any pairwise election with any other candidate, A will win. So regardless of where A appears on the agenda, A will beat the winner of all the previous contests on the agenda (if there are any), and then A will go on to beat every candidate listed after A on the agenda (if there are any). So at the end, A is the winner.

Now for the alternative. If A is not a Condorcet winner, then A loses at least one pairwise election. So say B wins a pairwise election against A. Construct an agenda so that A appears first and B appears second; place the other candidates after that in any order. Then in the first election, A will be eliminated and thus A loses the sequential pairwise voting with this agenda.