When do Prices Coordinate Markets?

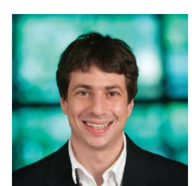
Ryan Rogers

Penn

Joint work with:

Justin Hsu, Jamie Morgenstern, Aaron Roth, and Rakesh Vohra






Prices are remarkable

- Markets are decentralized.
- Individuals observe prices, and buy bundles of goods that optimize their own utility functions. And somehow:
  - Markets clear! No substantial shortages or surpluses
  - The resulting allocation is pretty good.

What does the theory tell us?

- A commodity market is defined by:
  - A set of m types of discrete *goods* $g \in G$, each with *supply* $s_g \geq 1$
  - A set of n *buyers* $i \in N$, each with a valuation function $v_i: 2^G \rightarrow [0,1]$
- A feasible allocation is a function $\mu: N \rightarrow 2^G$ such that:
  - For each g$\in G$: $|\{i \in N: g \in \mu(i)\}| \leq s_g$
- The optimal welfare in a market is:

$$OPT = \max_{\mu} \sum_{i=1}^{n} v_i(\mu(i))$$

What does the theory tell us?

- When facing prices, people are self interested.
  - The *demand set* $D_i(p)$ for a bidder i at prices p is the set of bundles that he values maximally, given the prices:

$$D_i(p) = \arg\max_{S \subseteq G} \left(v_i(S) - \sum_{g \in S} p_g \right)$$

  - Buyers will always choose to buy a bundle in their demand set.

What does the theory tell us?

- A set of prices $p = (p_1, \dots, p_m)$ are Walrasian equilibrium prices if there exists a feasible allocation μ such that :
  - For every buyer $i \in N$, $\mu(i) \in D_i(p)$, and
  - $|\{i : g \in \mu(i)\}| < s_g$ only if $p_g = 0$
- For such a μ, we say (p, μ) form a Walrasian equilibrium.

What does the theory tell us?

- Some remarkable facts:
  - If (p, μ) is a Walrasian equilibrium, then $\sum_i v_i(\mu(i)) = OPT$ (First Welfare Theorem)
  - If buyer valuations satisfy the *gross substitutes property*, then Walrasian equilibria are guaranteed to exist! [KC82,GS99]
  - In fact, natural ascending price auction dynamics converge to them!
  - In fact, many such prices – the set of Walrasian equilibrium prices forms a lattice. The ***minimal*** Walrasian equilibrium prices are focal: many tatonnemont processes converge to them, and they correspond to VCG prices for unit-demand valuations.

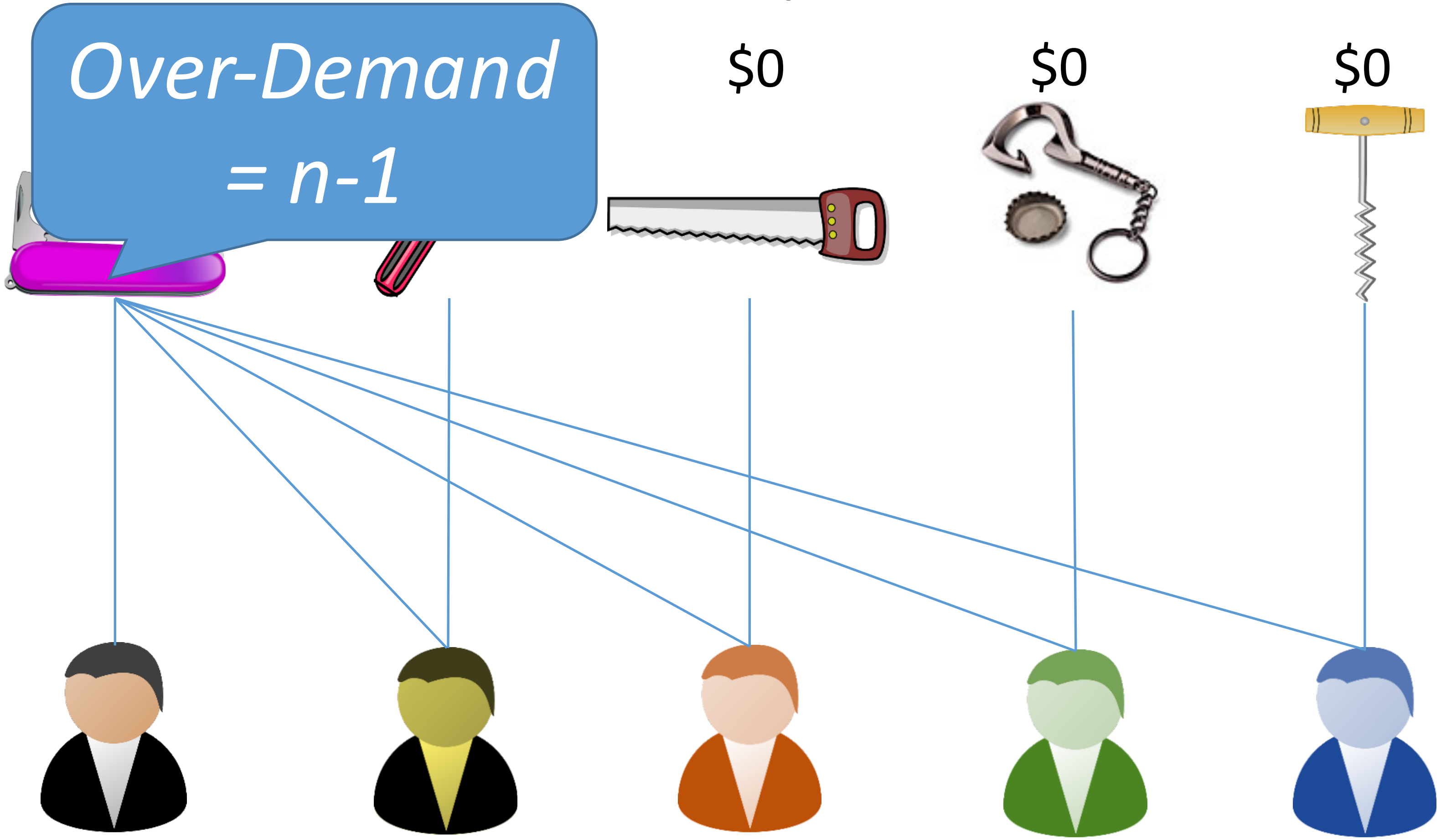
Interpretation?

“Prices *optimally* coordinate a large class of markets.”

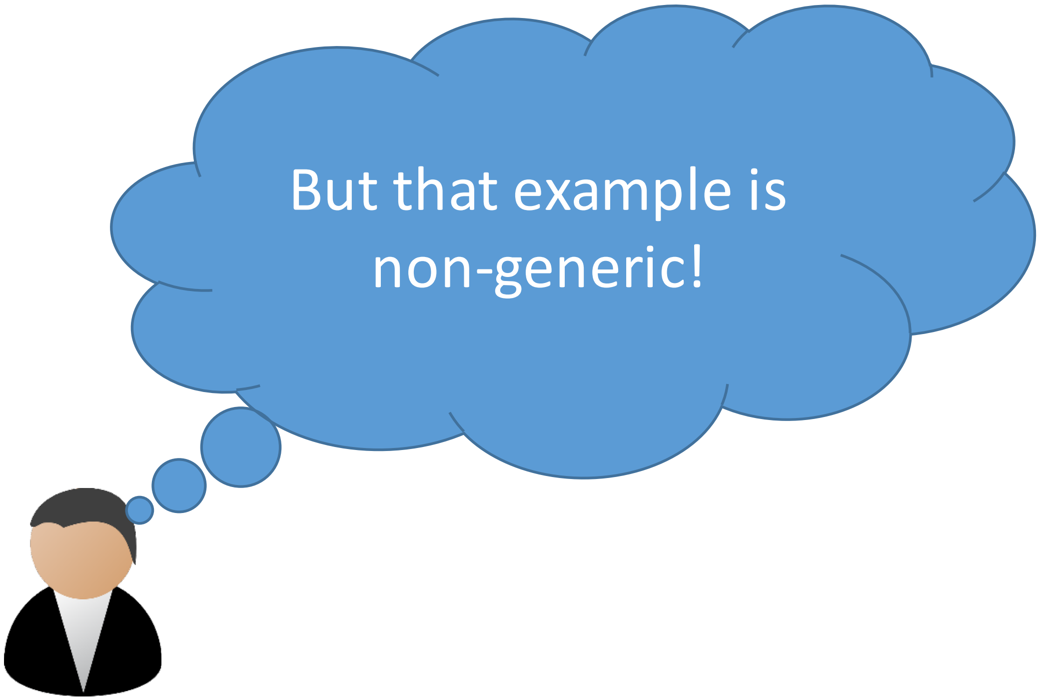
Some problems with that interpretation

- No coordination?
  - Each bidder i may have many bundles in his demand set $D_i(p)$, and to obtain the optimal allocation μ, these ties have to be broken in a coordinated manner.

The coordination problem



The coordination problem



The coordination problem

- There will *always* be over-demand at the minimal Walrasian equilibrium prices!
  - Suppose otherwise: $Let\ p(g) > 0\ and\ OD(g) = 0$
    - Must have $|\{i \in N : g \in \mu(i)\}| = s_g$
    - All buyers *j* such that $g \notin \mu(j)$ cannot demand any bundle with *g*, otherwise *OD(g)>0*.
    - All the other buyers strictly prefer a bundle that does not contain *g.*
    - Then we could take any nonzero price and lower it by ϵ
    - The prices would remain Walrasian, contradicting minimality.

The best we can hope for is approximation.

Some problems with that interpretation

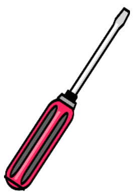
- Where do the prices come from?
  - There are many natural interactive auction procedures that converge to Walrasian prices.
  - But in practice, we face fixed prices and do not engage in auctions.

Where do prices come from?

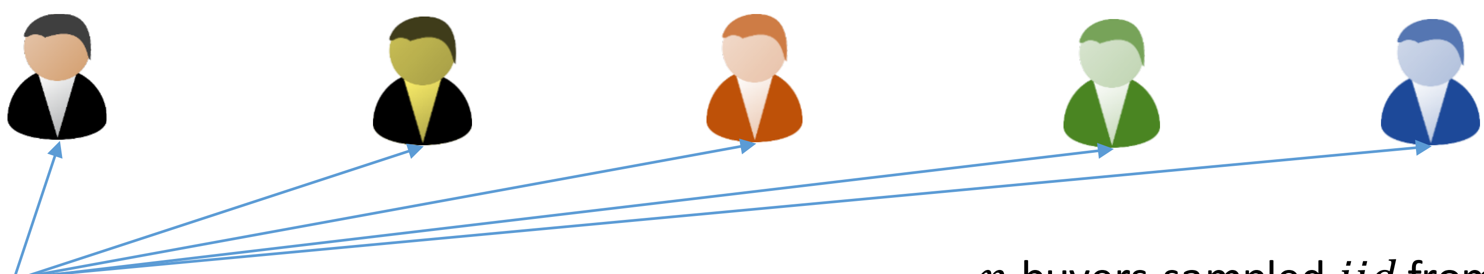
- In practice, prices encode “distributional information”.
- But:
  - How much distributional information is necessary?
  - Implies we are facing approximate equilibrium prices – only exacerbates coordination.

This work

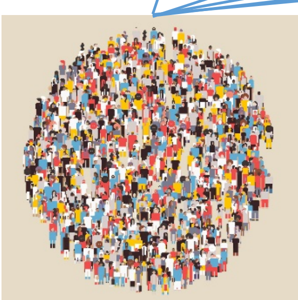
$1.25 $5.99 $0.50 Minimal equilibrium prices



m types of goods g with supply s_g

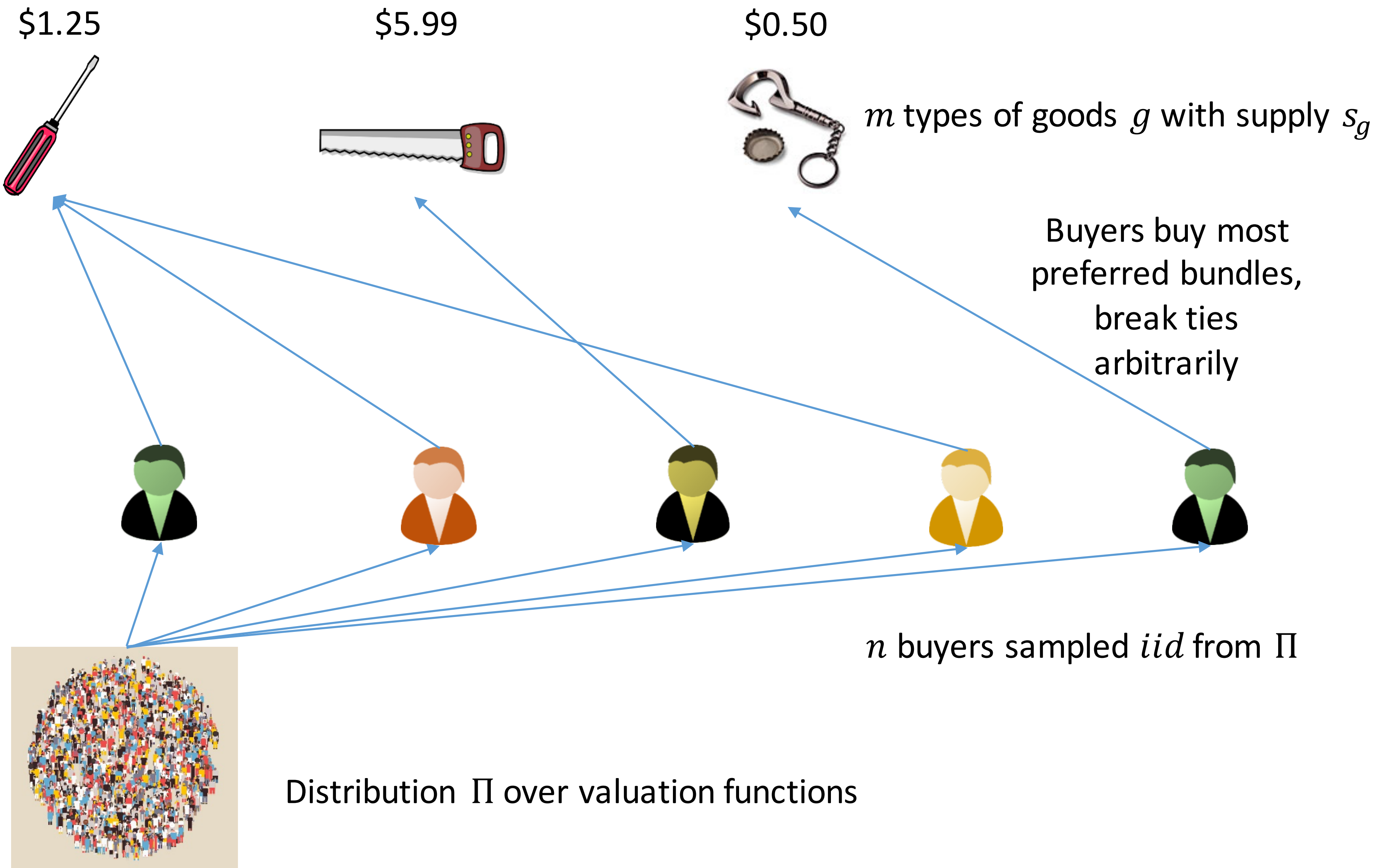


n buyers sampled iid from Π



Distribution Π over valuation functions

This work


$1.25
$5.99
$0.50
m types of goods g with supply s_g
Buyers buy most preferred bundles, break ties arbitrarily
n buyers sampled iid from Π
Distribution Π over valuation functions

This work

- How much over-demand do we expect? How high is welfare?
- What do these things depend on?
  - Particulars of the distribution Π?
  - Complexity of valuation functions v_i?
  - Number of buyers n?
  - Number of goods m?

This work

Training Error

- First question: When do the *exact* minimal equilibrium prices induce little over-demand?
  - Remember: Not always. And we must always have some over-demand.

Testing Error

- Second question: How well do equilibrium prices computed on a sample generalize to new buyers drawn from the same distribution?
  - And how much data is needed?

First Question

- Warmup: Unit demand buyers
  - $v_i(S) = \max_{g \in S} v_i(\{g\}) \equiv \max_{g \in S} v_{i,g}$
  - (i.e. buyers just want 1 item)
- Genericity Assumption:
  $\sum_{i,g} a_{i,g} v_{i,g} = 0$ with $a_{i,g} \in \{-1,0,1\}$ iff $a_{i,g} = 0$ for all i, g.
  (i.e. valuations are linearly independent over $\{-1,0,1\}$)

Notes on assumption:

- Implies the welfare-optimal allocation μ is unique.
- Satisfied with probability 1 for any continuous perturbation of valuations.

Over-Demand

- Theorem: If valuations satisfy our genericity assumption, and p are the minimal Walrasian equilibrium prices, then for every good g:

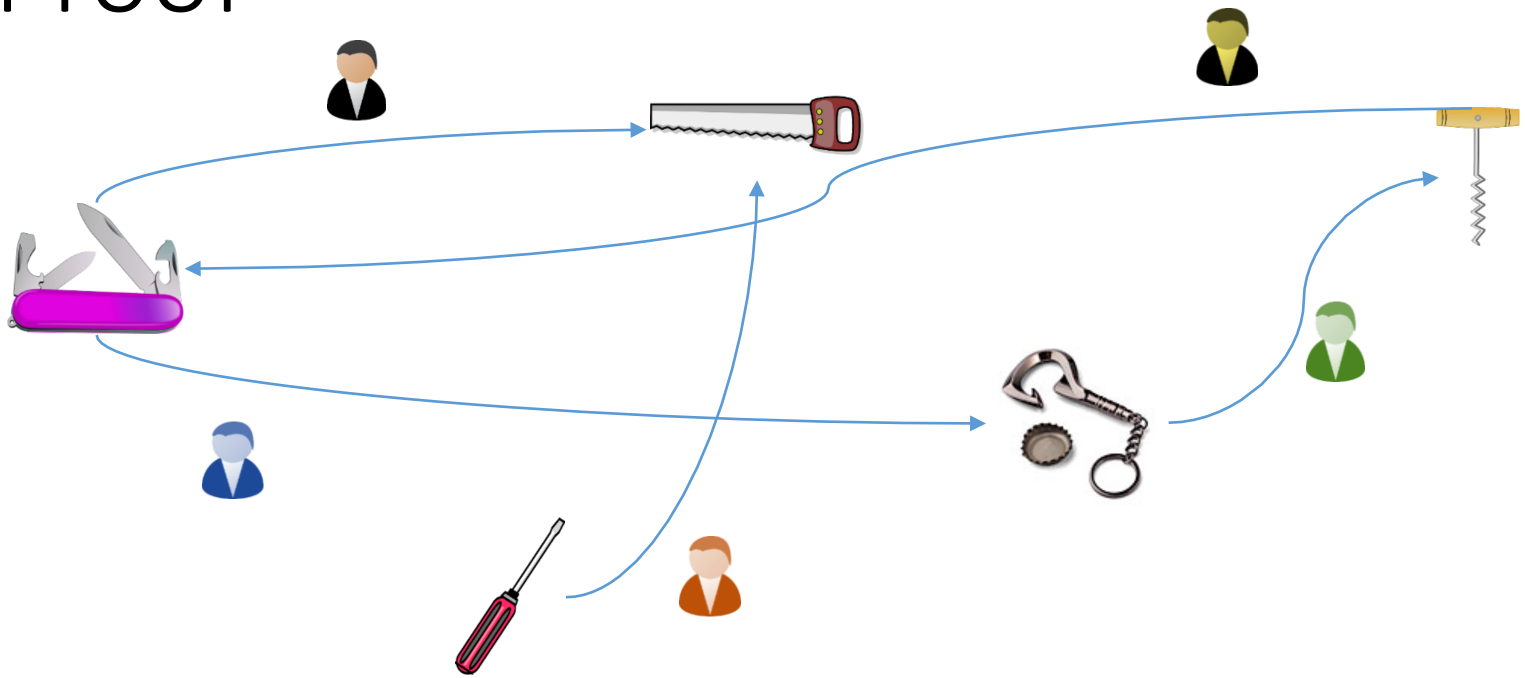
$$|\{i \in N: g \in D_i(p)\}| \leq s_g + 1$$

(no matter how people break ties, over-demand on any good is at most 1)

Proof

- Fix the optimal allocation μ, minimal equilibrium prices p.
- Construct a graph $G = (V, E)$ where:
  - $V = \{1, \dots, m\}$ – vertices are types of goods.
  - $(g, g') \in E$ for every buyer i with $\mu(i) = g$, $g' \in D_i(p)$ and $g' \neq g$

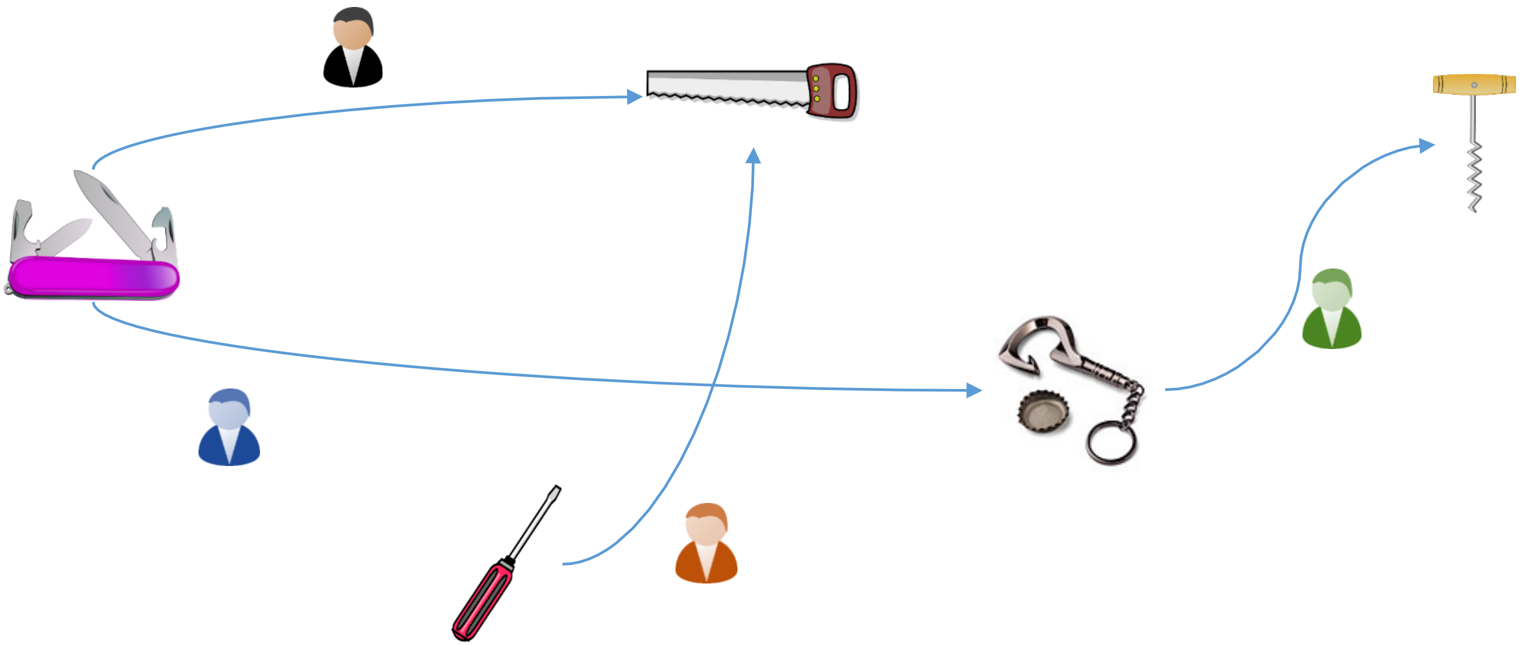
Proof



Claim 1: The graph must be acyclic.

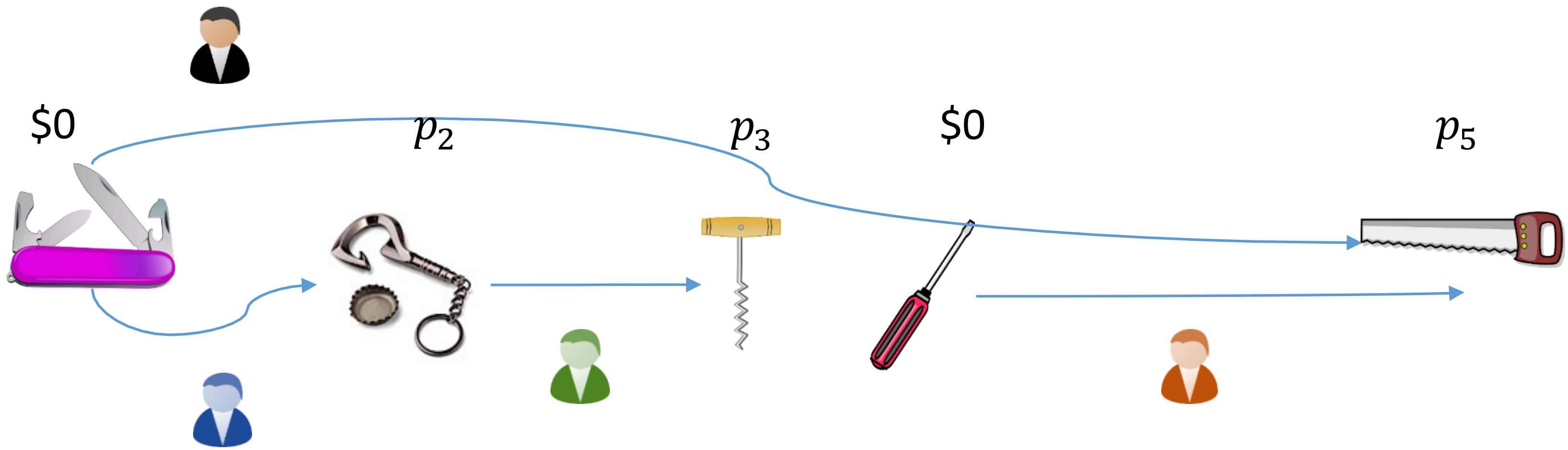
Proof: Otherwise players could swap allocations around a cycle, and arrive at a distinct max-welfare allocation μ'.

Proof



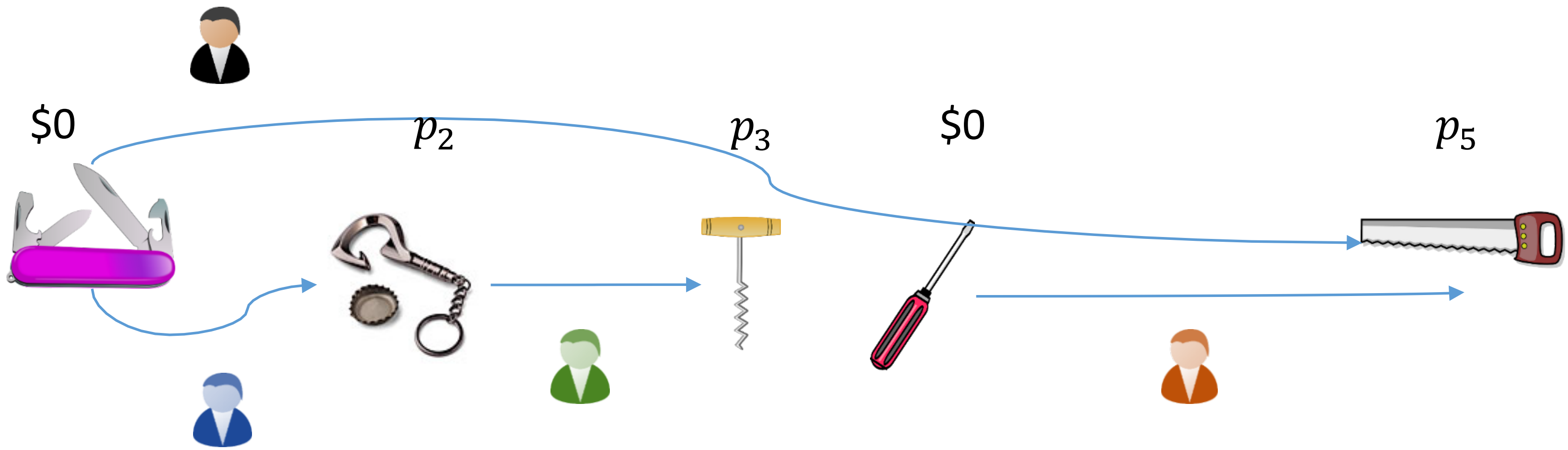
So, we can topologically sort the graph, rename vertices in order.
First vertex has indegree zero.

Proof



Claim 2: Any good with in-degree zero must have price 0
Proof: Otherwise we could lower the price, contradicting minimality.

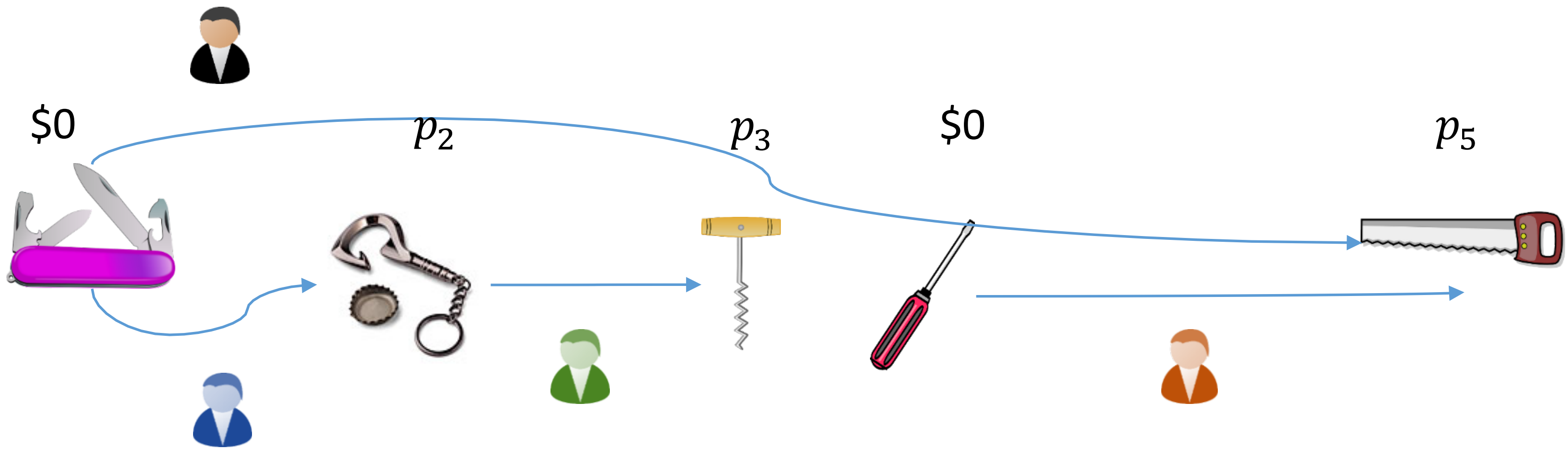
Proof



Claim 3: All prices p_g can be written as linear combinations of valuations $v_{i,h}$ where $h \leq g$ and coefficients are in $\{-1,0,1\}$.

Proof: Base case: $p_1 = 0$

Proof



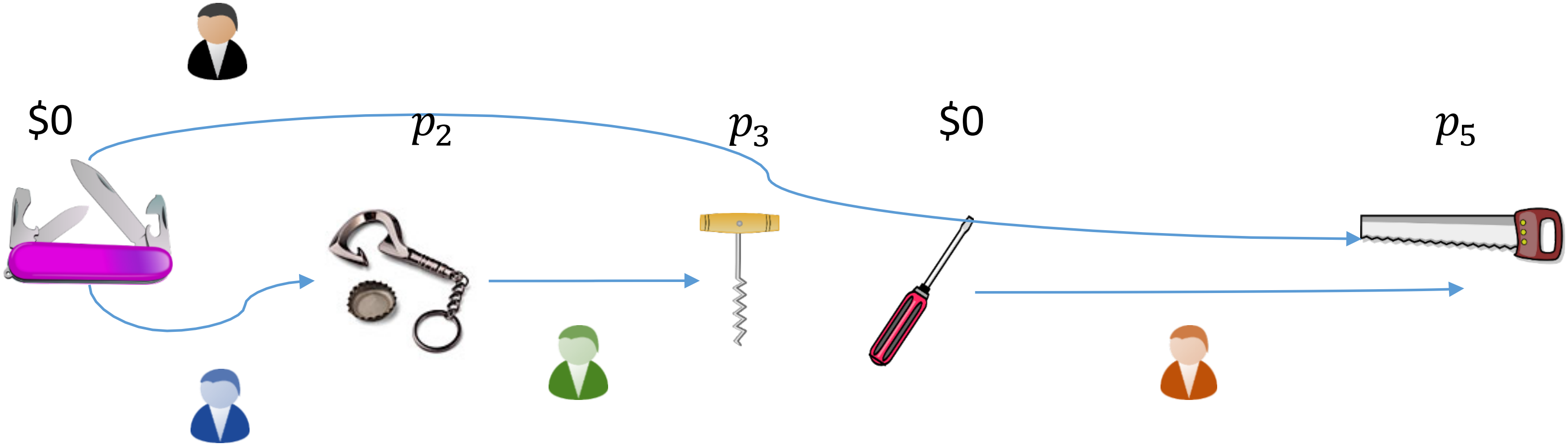
Claim 3: All prices p_g can be written as linear combinations of valuations $v_{i,h}$ where $h \leq g$ and coefficients are in $\{-1,0,1\}$.

Proof: Inductive case: If g has positive in-degree, there is a buyer i with $\mu(i) = g'$ for $g' < g$, and $g \in D_i(p)$. i.e:

$$v_{i,g'} - p_{g'} = v_{i,g} - p_g \text{ or:}$$

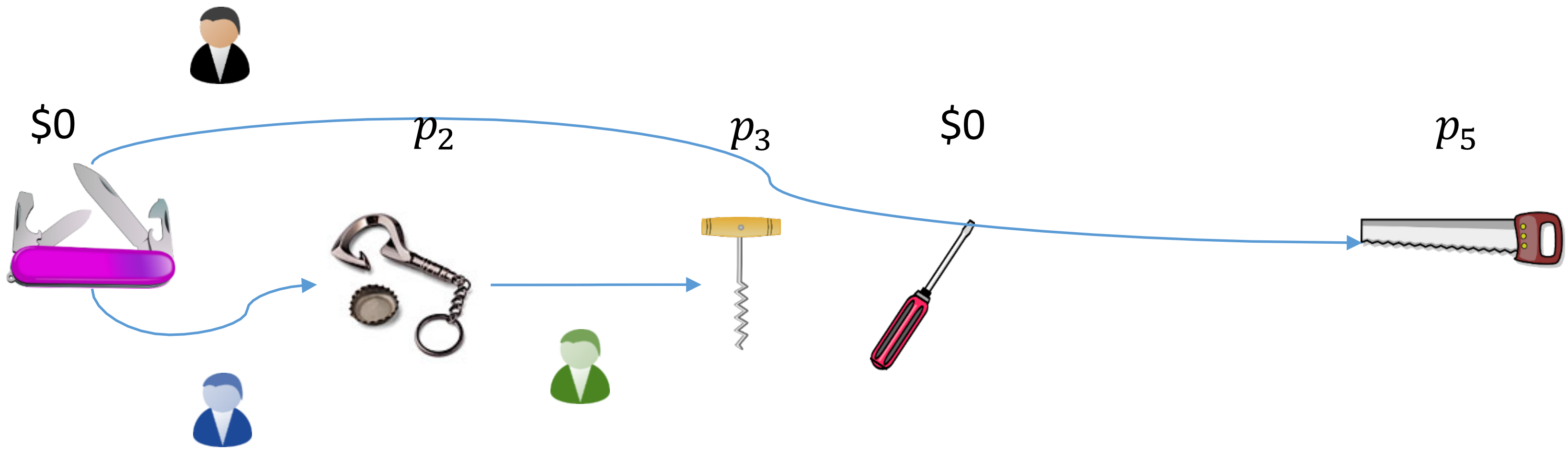
$$p_g = v_{i,g} - v_{i,g'} + p_{g'}$$

Proof



Claim 4: All goods g have in degree ≤ 1

Proof



Claim 4: All goods g have in degree ≤ 1

Proof: Suppose not. Then there are two buyers $i \neq i'$ with:

$$v_{i,\mu(i)} - p_{\mu(i)} = v_{i,g} - p_g \text{ and } v_{i\prime,\mu(i\prime)} - p_{\mu(i\prime)} = v_{i\prime,g} - p_g$$

This gives us two expressions for p_g. Subtracting them:

$$v_{i,g} - v_{i',g} + v_{i,\mu(i)} - v_{i',\mu(i')} + p_{\mu(i)} - p_{\mu(i')} = 0$$

The coefficients aren't all zero since $\mu(i), \mu(i') < g$

Contradicts genericity!

Welfare

- We have shown that over-demand is low when buyers grab any good from their demand set.
- Impose the following rule: If a buyer is indifferent to empty allocation or getting a good, then she takes a good.
- If buyers grab demanded goods $\{b_1, \ldots, b_n\}$ following the rule above, then the resulting welfare is close to optimal:
  - $Welfare(\{b_1, \ldots, b_n\}) \geq OPT - 2m$

Extending Result

Over-Demand and Welfare Results not limited to unit demand valuations!

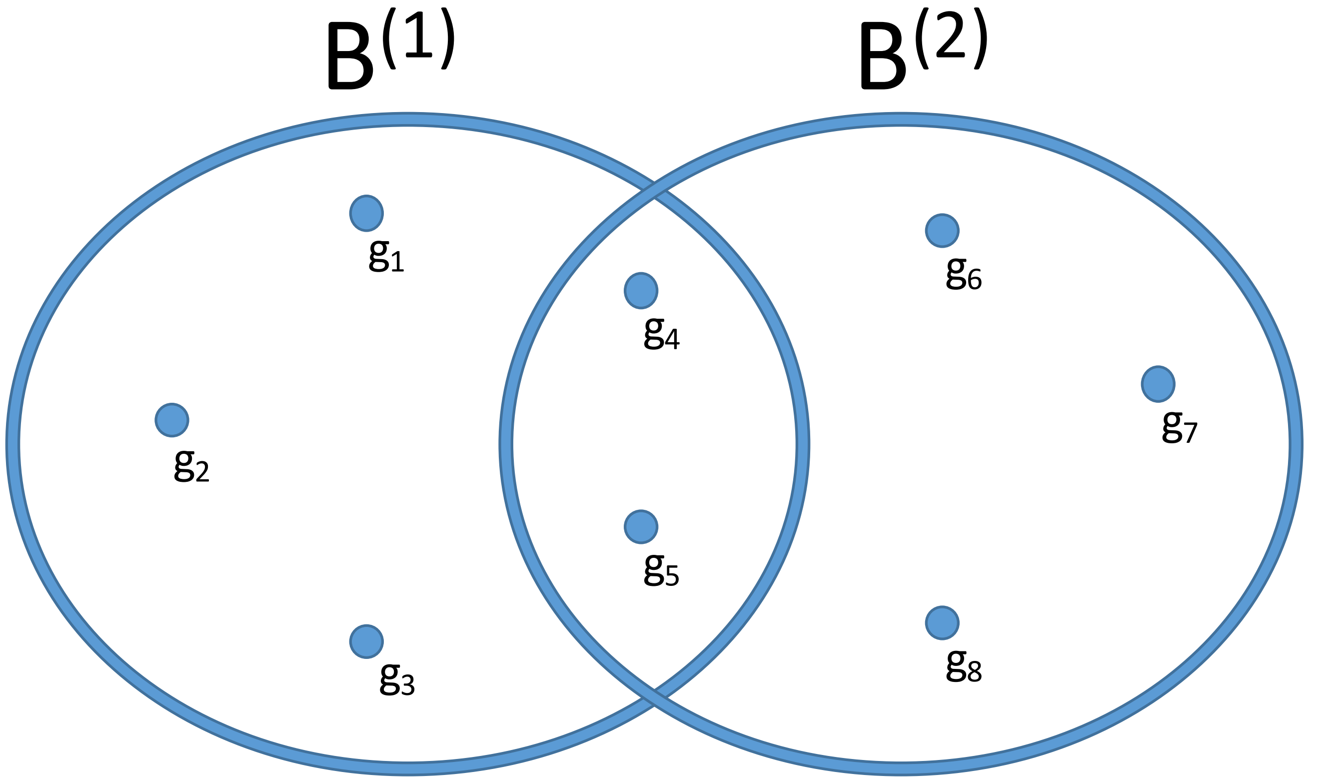
We now consider buyers getting bundles of goods.

Gross Substitutes Prelims

- A valuation v: $2^G \rightarrow [0,1]$ obeys GS if for all price vectors $p' \geq p$, and $S \in D(p)$ there exists a bundle $S' \in D(p')$ such that
  - $S \cap \{g: p(g) = p'(g)\} \subseteq S'$

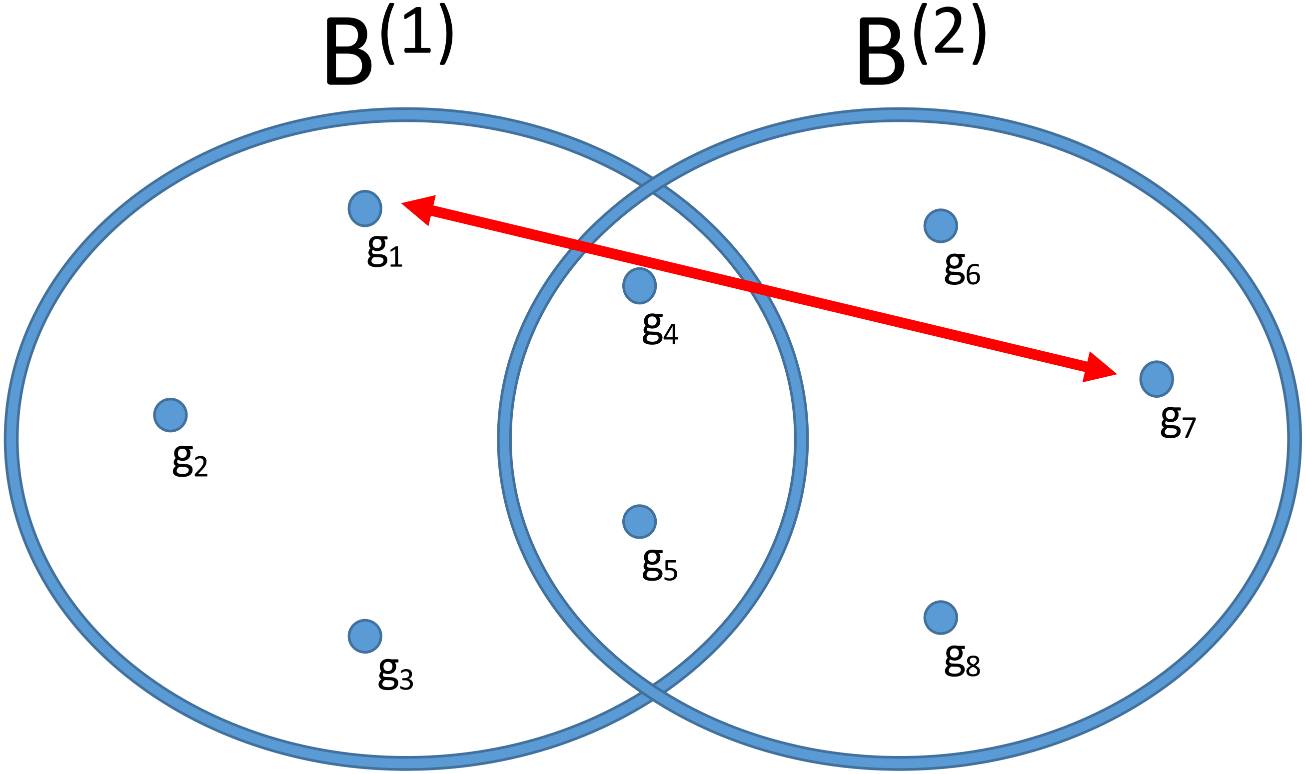
- It is known that for GS valuation *v*, the collection of ***minimal demand sets***
  $$D^*(p) = \{S \in D(p): S' \notin D(p), \forall S' \subset S\}$$
  forms the bases of a matroid.

Exchange Property for Matroids



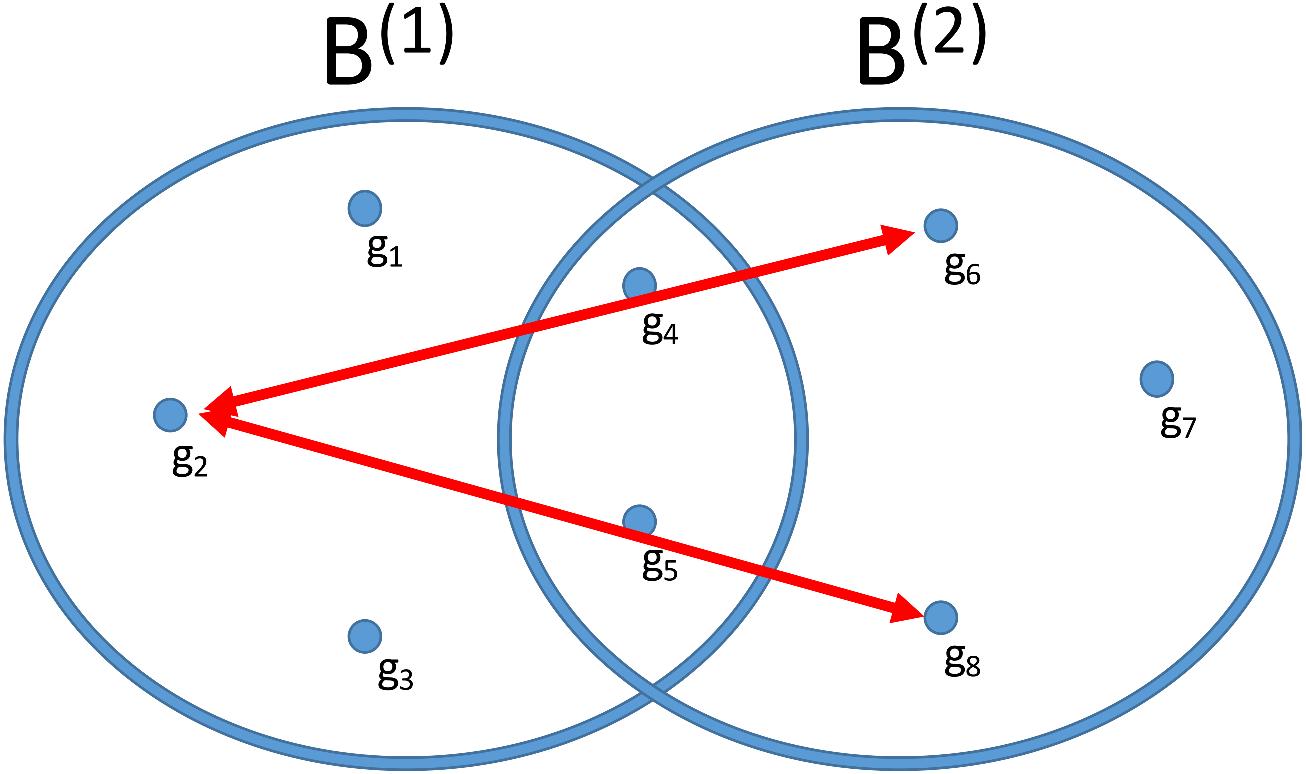
Exchange Property for Matroids

$B^{(1)} \cup g_7 \setminus g_1 \in D_i^*(p)$



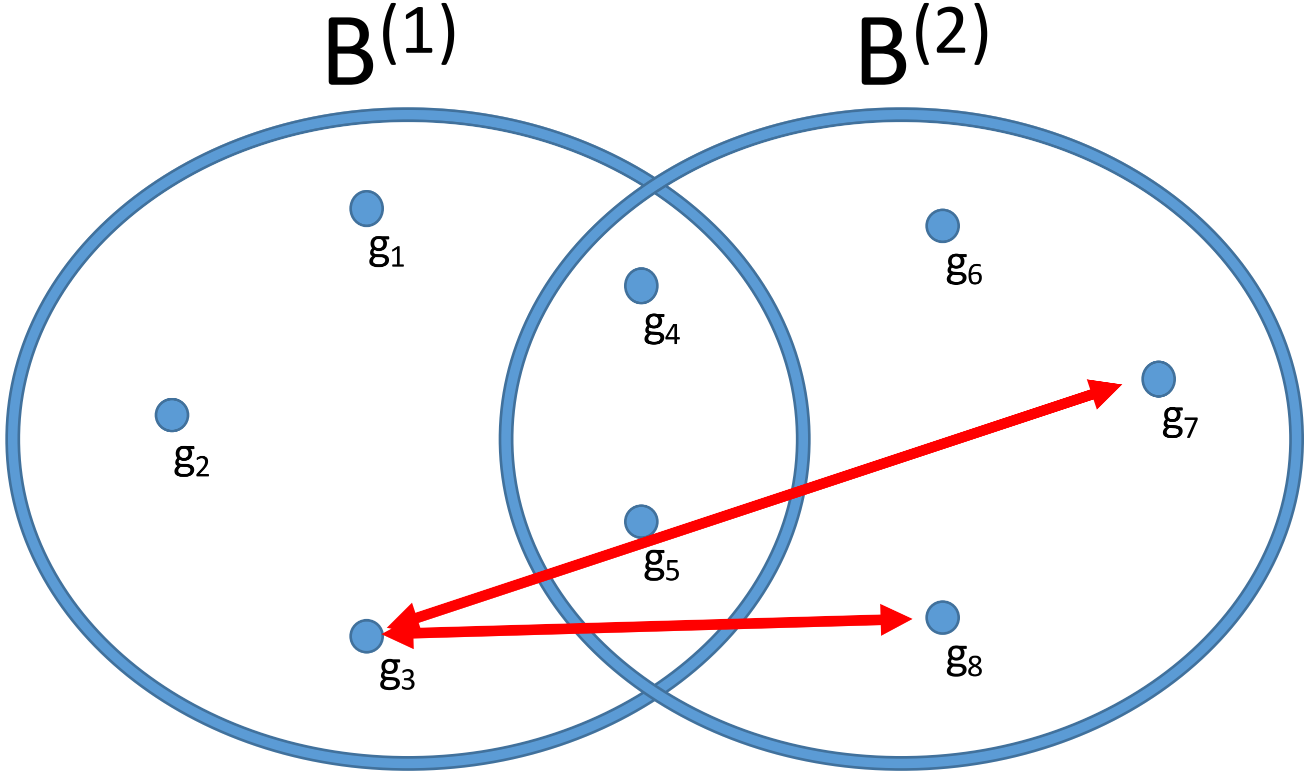
Exchange Property for Matroids

$B^{(1)} \cup g_6 \setminus g_2 \in D_i^*(p)$ $B^{(1)} \cup g_8 \setminus g_2 \in D_i^*(p)$



Exchange Property for Matroids

$B^{(1)} \cup g_7 \setminus g_3 \in D_i^*(p)$ $B^{(1)} \cup g_8 \setminus g_3 \in D_i^*(p)$



Swap Graph for GS

- Let $\{v_i\}$ be GS valuations for all n buyers. Fix Walrasian equilibrium (p, μ) with minimal prices, and minimum demand sets $M_1, \dots, M_n$ where $M_i \subseteq \mu(i)$.
- Have a node for every good in G
- There is an edge (a, b) for every buyer i where $a \in M_i$ and $b \notin \mu(i)$ and there exists a $B \in D_i^*(p)$ with $b \in B$ and

$$M_i \cup b \setminus a \in D_i^*(p)$$

Swap Graph for GS

- What if buyers are indifferent to bundles of different sizes?
- Include a null node $\bot$
- If $p(b) > 0$, there is an edge from $\bot$ to b for each buyer i that has minimum demand set $B_i \in D_i^*(p)$ and $B_i \cup b \in D_i(p) \setminus D_i^*(p)$

Proof Outline for Over-Demand

- Define a genericity condition for GS valuations
- Show the swap graph is acylic
- Show source nodes in the swap graph have price zero
- Show that the prices of goods can be written as an *integer* linear combination of “weights” from previous goods in a topological sort of the nodes.
- Bound in-degree.

Matroid Based Valuations

- Because the definition of Gross Substitutes is axiomatic rather than constructive, it is not clear if any GS valuation satisfies some generic condition.
- ***Matroid Based Valuations*** gives a constructive way to define valuations that are contained in GS.
- Conjectured by [Ostrovsky, Paes-Leme 15] that *MBV* is equal to *GS.*

Matroid Based Valuations

- A valuation v is in VIWM if there exists a matroid $M = (I, G)$ and weights $\{w_g\}$ such that
  - $v(S) = max_{T \subseteq S, T \in I} \sum_{g \in T} w_g$
- Endowment operation: $v(S) = v'(S \cup J) - v'(J)$ where $T \cap J = \emptyset$.
- Merge opetation: $v(S) = \max_{(S_1, S_2) = S} v_1(S_1) + v_2(S_2)$.
- MBV is the smallest class of valuations that contain VIWM and is closed under endowment and merge operations.

Generic MBV

- The collection of buyers valuations $\{v_i\}$ are GMBV if they are MBV and all weights W for all the buyers and all the goods are linearly independent over the integers

$$\sum_{w \in W} \alpha_w w = 0 \quad \text{for } \{\alpha_w\} \in \mathbb{Z}$$
$$\text{iff} \quad \alpha_w = 0 \text{ for every } w \in W$$

Over-Demand for GMBV

- We can show that for buyers with GMBVs that take only minimum demand bundles at the minimal Walrasian prices, then the over-demand for any good is at most 1 – ignores the use of the null node and is bad for bounding welfare.
- The non-degenerate correspondence for buyer *i* is,
  - $\widehat{D_i}(p) = \{S \in D_i(p): v_i(S \setminus g) < v_i(S),\ \forall g \in S\}$
- If buyers take bundles in their non-degenerate correpondence, then over-demand is at most 1.

Welfare for GMBV

- We bound the over-demand for buyers that take any non-degenerate bundle by 1
- If buyers take max cardinality non-degenerate bundles, $\{B_1, \ldots, B_n\}$, then the Welfare is close to optimal:

$$Welfare(B_1, \ldots, B_n) \geq OPT - 2m$$

Second Question

- We have a condition under which (exact) equilibrium prices computed on a population induce low over-demand.
- How well does this generalize if we use the same prices on a new population?

Over-Demand Generalization

Punchline:

On a fresh sample of buyers, the demand for any good *g* satisfies:

$$\left| |\{i : g \in D_i(p)\}| - s_g \right| \leq O\left(\sqrt{s_g \cdot m \cdot \log\frac{m}{\delta}}\right)$$

with probability $(1 - \delta)$.

i.e. the total demand for any good is w.h.p. within a $(1 + \epsilon)$ factor of the supply whenever:

$$s_g \geq \tilde{O}\left(\frac{m}{\epsilon^2}\right)$$

Proof Outline for Generalization

- Fix a tie breaking rule agents i use to select bundles $S_{v_i}(p) \in D_{v_i}(p)$ given prices p.
- For a fixed $p \in \mathbb{R}^m$ define $f_p(v_i) = S_{v_i}(\mathrm{p})$ and for each $g \in G$ define $d_p^g(v_i) = \begin{cases} 1 \text{ if } g \in S_{v_i}(p) \\ 0 \text{ otherwise} \end{cases}$
- Given a distribution Π over valuation functions, the expected demand for g given prices p is:

$$n \cdot \mathop{\mathrm{E}}_{v_i \sim \Pi}\left[d_p^g(v_i)\right]$$

Uniform Convergence

- It suffices to obtain uniform convergence of the empirical averages over the sets of functions:

$$C^g = \{d_p^g : p \in \mathbb{R}^m\}$$

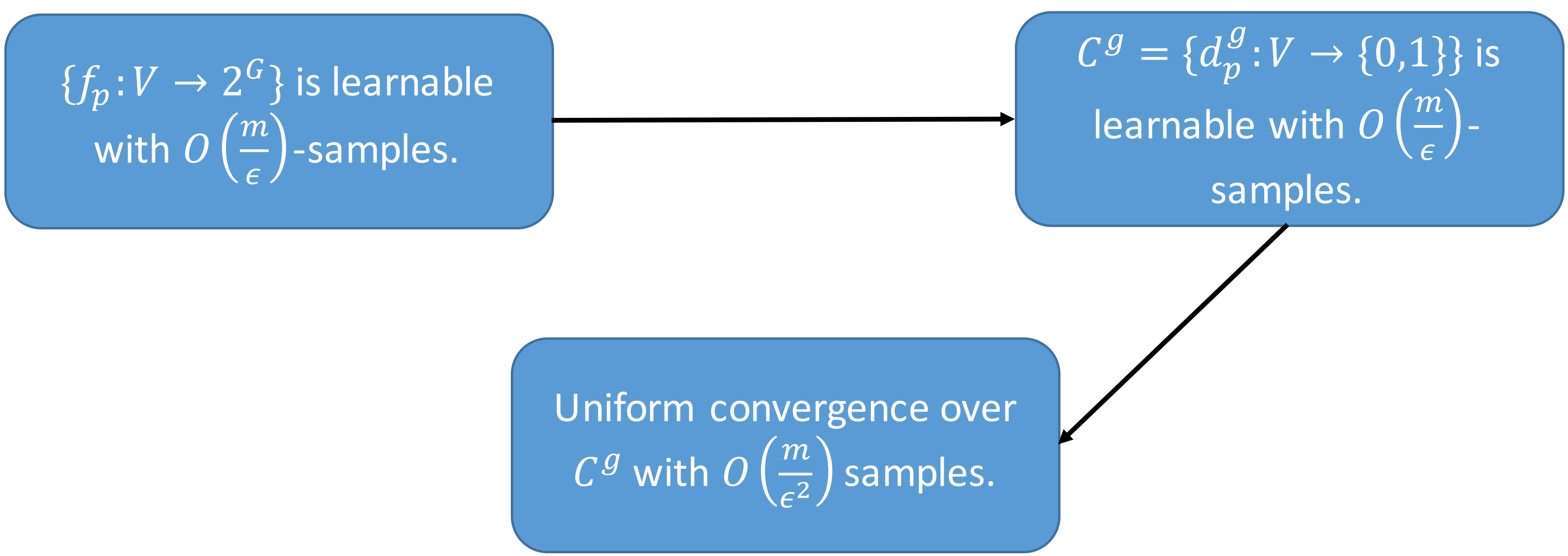
to their expectations.

(then, with high probability over the draws of two samples of bidders, for *every* price vector, demand is similar on both samples.)

(In particular, for the Walrasian prices computed on the first sample, for which we know over-demand is small)

Learning and Uniform Convergence

- For one dimensional, Boolean learning problems, learning over C^g $\Leftrightarrow$ Uniform convergence over C^g.
  - But not so for multi-dimensional/real valued learning problems.



Welfare Generalization

Punchline:

The welfare induced by the chosen price vector on a new sample of buyers is with high probability at least:

$$(1-\epsilon)\cdot OPT$$

Whenever $OPT \geq \tilde{O}\left(\frac{m^4\sqrt{n}}{\epsilon^2}\right)$

Outline of Proof

- Welfare is a real valued function
- Directly bound the Pseudo-Dimension via shattering.
- Want to bound the following: For *n* buyers, how many distinct allocations can be induced by varying over all price vectors?
- We can bound this by $2^{O(m^2)}$, so that Pseudo-Dimension is no more than $\tilde{O}\ (m^2)$.

Do prices coordinate markets?

Generically, they do!

THANKS