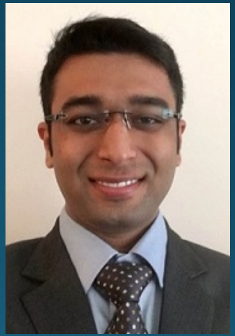
Max-Information, Differential Privacy, and Post-Selection Hypothesis Testing

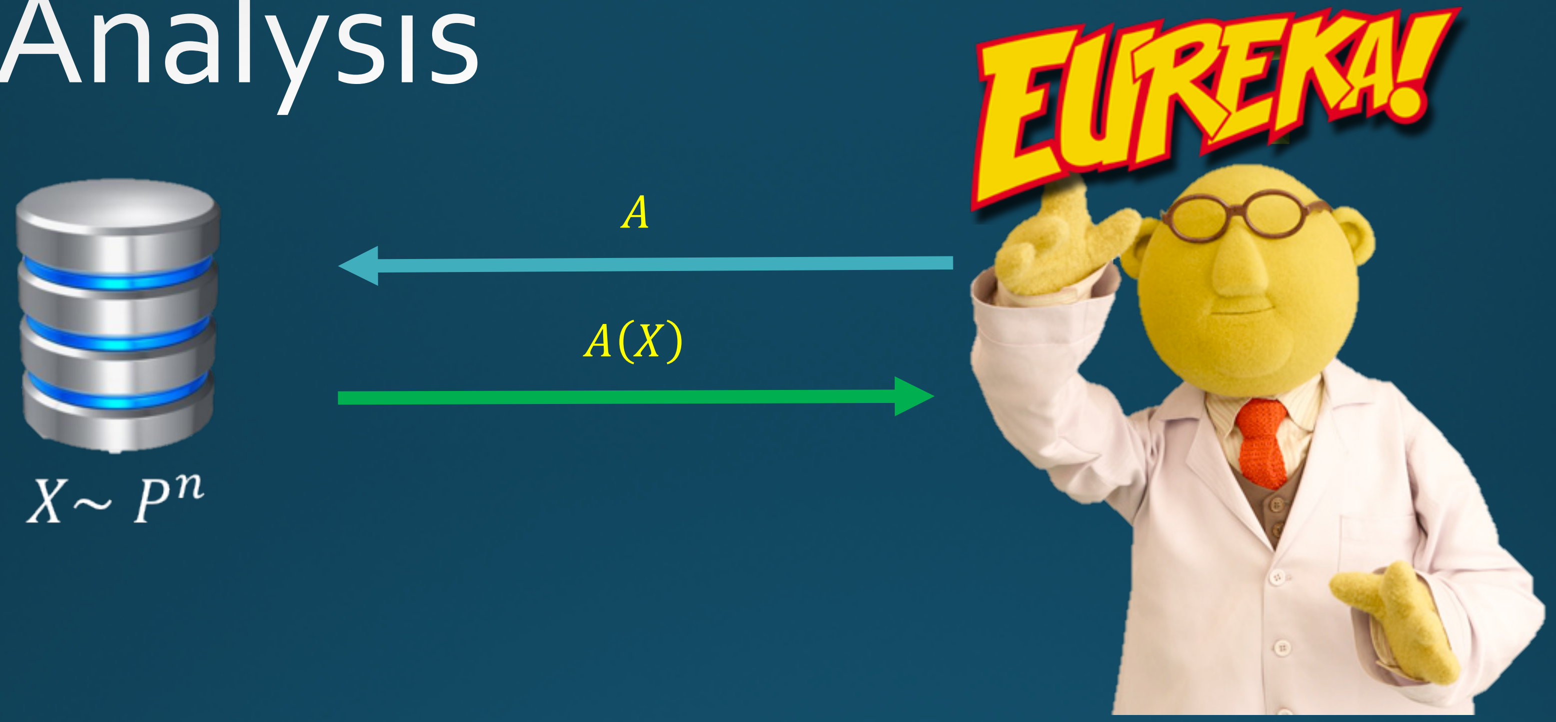
Ryan Rogers, Aaron Roth, Adam Smith, and Om Thakkar



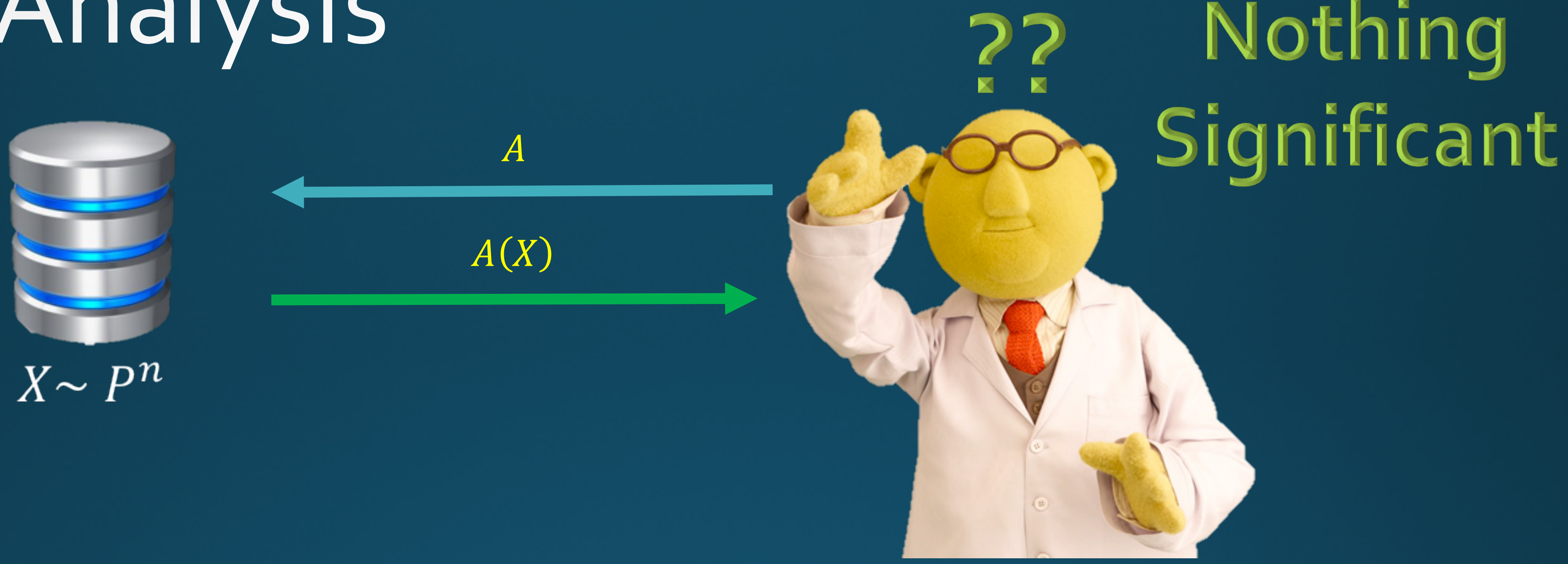
Supported by grants from the Sloan Foundation and NSF:
CNS-1253345, CNS-1513694, IIS-1447700.

Data Analysis



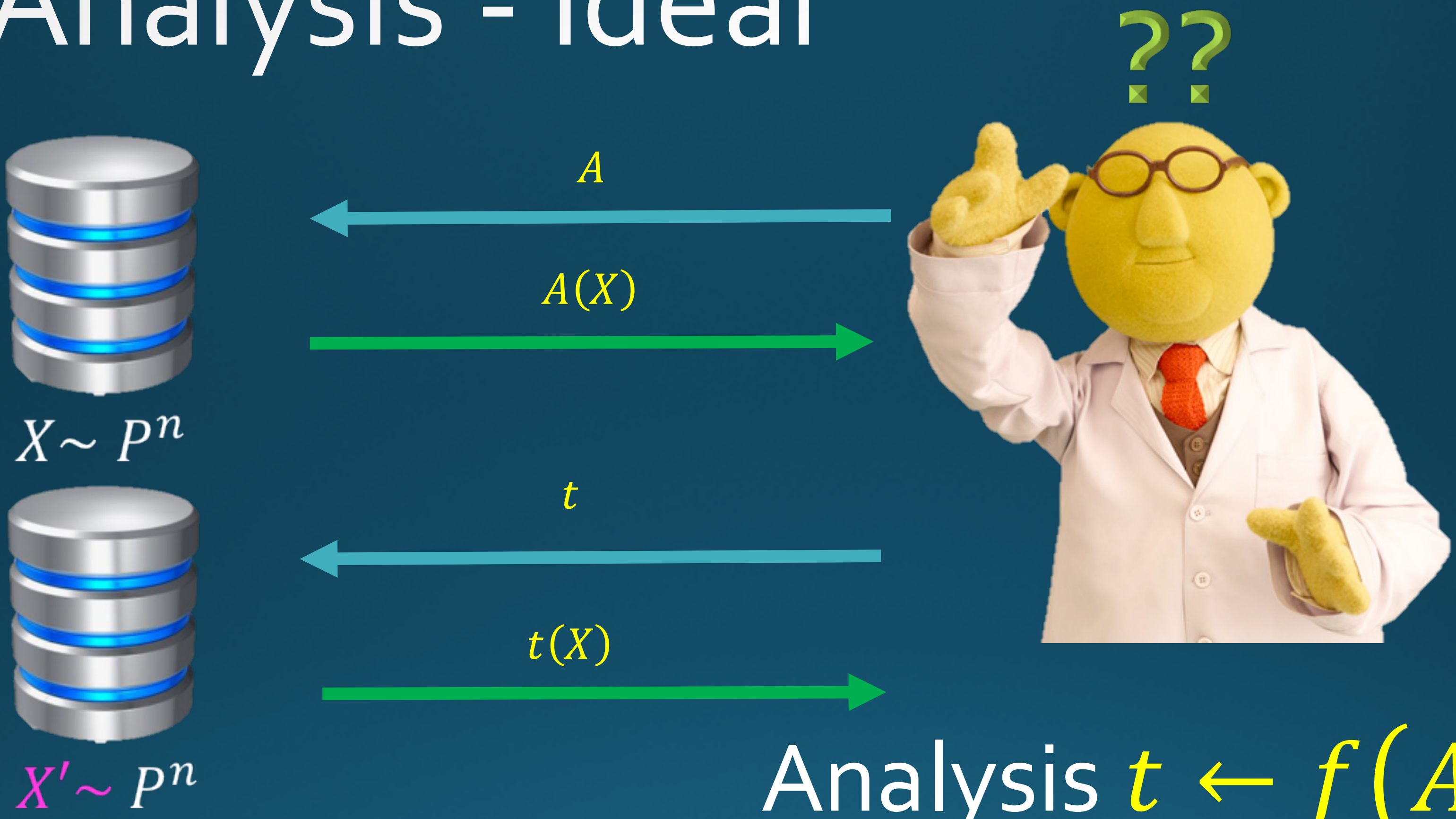
Analysis A

Data Analysis



Analysis $t \leftarrow f\big(A(X)\big)$

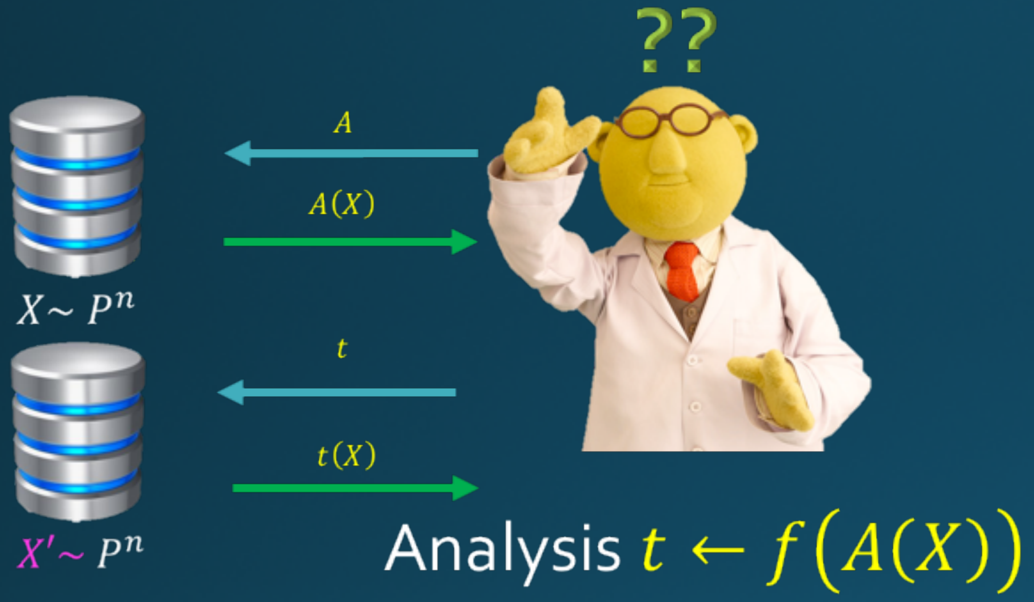
Data Analysis - Ideal



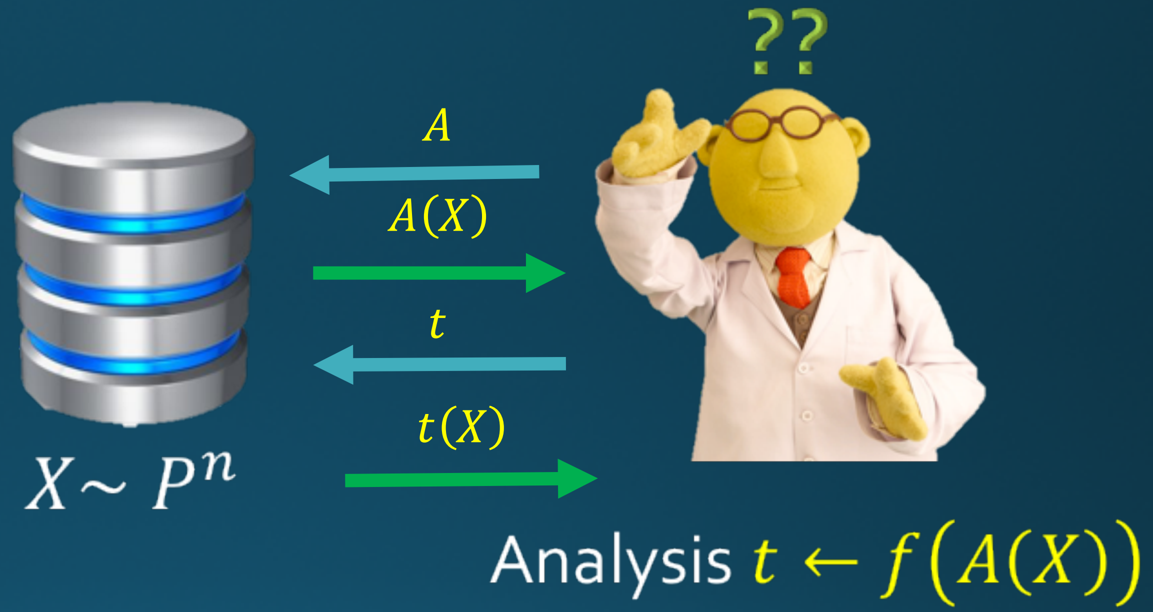
Analysis $t \leftarrow f\big(A(X)\big)$

A lot of existing theory assumes tests are selected independently of the data.

Ideal World



Real World



How can we provide statistically valid answers to adaptively chosen analyses?

Ideal World

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The Replication Crisis in Psychology
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3 COMMENTS
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Article
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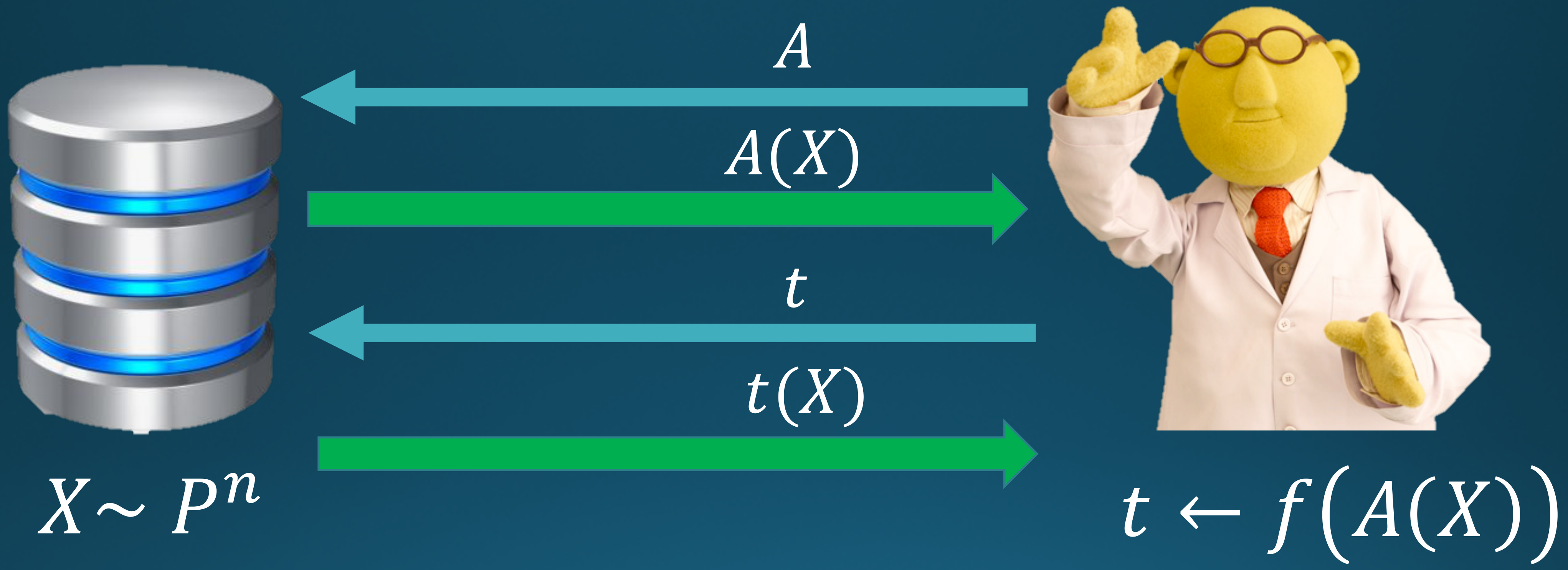
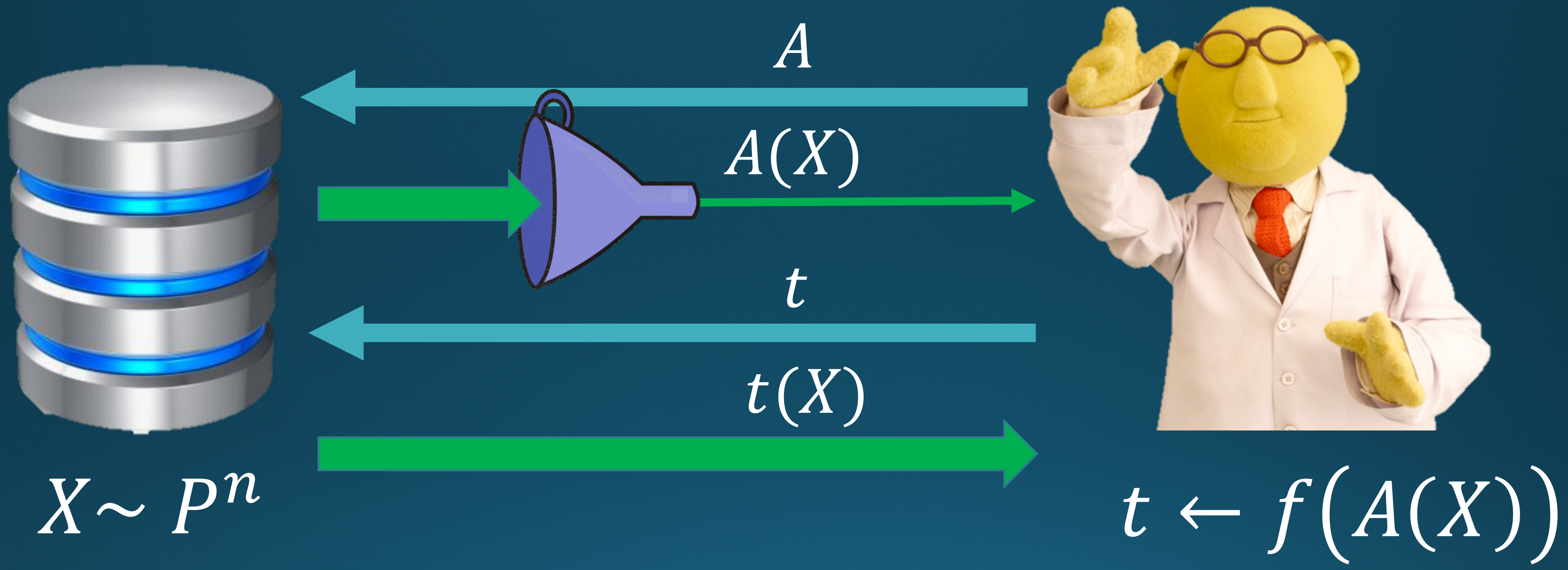
Abstract
The Statistical Crisis in Science
Data-dependent analysis—a "garden of forking paths"— explains why many statistically significant comparisons don't hold up.
Andrew Gelman and Eric Loken
There is a growing realization that reported "statistically significant" claims in scientific
a short mathematics test when it is expressed in two different contexts, involving either healthcare or the
This multiple comparisons issue is well known in statistics and has been called "p-hacking" in an influential

Adaptive Data Analysis



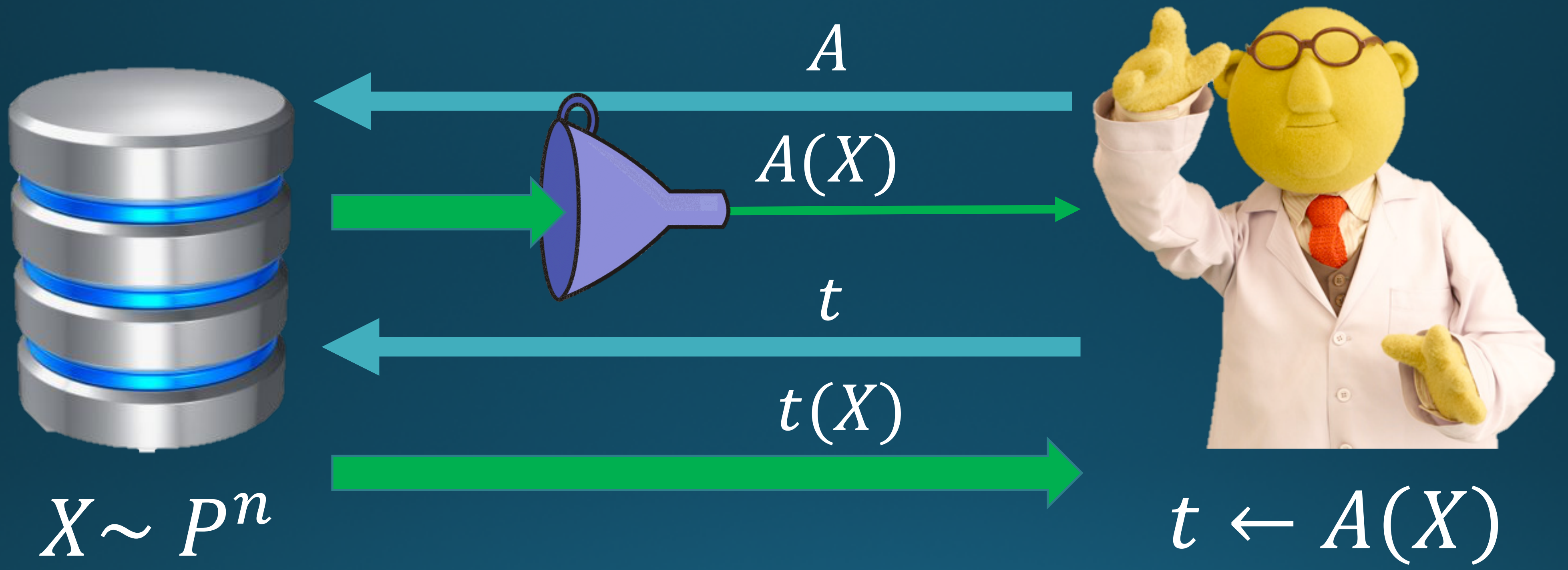
How can we provide statistically valid answers to adaptively chosen analyses?

Adaptive Data Analysis



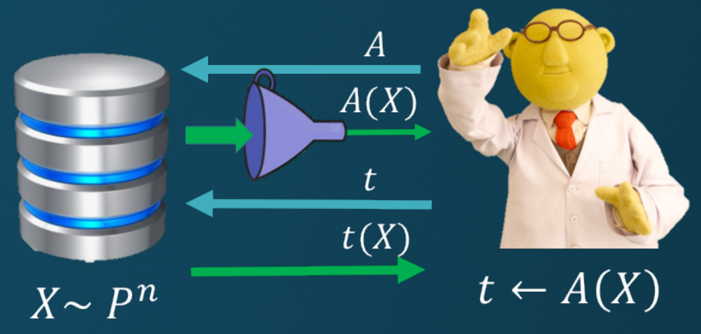
Answer: Limit the info learned about the dataset [Dwork,Feldman,Hardt,Pitassi,Reingold,Roth'15].

Adaptive Data Analysis



Answer: Limit the info learned about the dataset [Dwork,Feldman,Hardt,Pitassi,Reingold,Roth'15].

Contributions

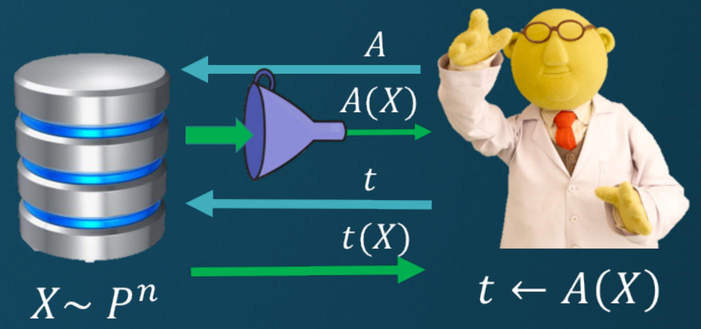


- Post-selection Hypothesis Testing
  - Bounded Max-Info $\implies$ Valid Tests
  - Tighter connection than previous results.

- Approximate Differential Privacy $\implies$ Bounded Max-Info
  - k rounds of adaptivity: max-info $\sim k$ rather than k^2.

Generalizes and unifies previous work

Related Work



- Lots of work in statistics community on post-selection inference [Freedman'83],[Leeb,Potscher'06],[Berk,Brown,Buja,Zhang,Zhao'13], …
  - Specific to type of analyses performed
- [DFHPRR](STOC'15,NIPS'15,Science'15)
  - Initial connections between information, privacy and adaptive analysis
- Accuracy for specific queries
  - [DFHPRR] (STOC'15,Science'15)
  - [Bassily,Nissim,Smith,Steinke,Stemmer,Ullman'16]
  - [Cummings,Ligett,Nissim,Roth,Wu'16]
  - [Russo,Zou'16]
  - [Wang,Lei,Fienberg'16]
- Impossibility results
  - [Hardt,Ullman'14], [Steinke,Ullman'15]

Hypothesis Testing

- Hypothesis test is defined by
  - null hypothesis $H_0 \subseteq \Delta(D)$ and
  - statistic:

$$t: D^n \to \{Inconclusive, Reject\}$$

- A *False Discovery* is when $X \sim P^n$ and $P \in H_0$ but $t(X) = Reject$

- Classical results apply when t is independent of X.
- Want to bound Pr[*False Discovery*] when $t \leftarrow A(X)$.

Max-Information [DFHPRR'15]

- Algorithm A has small max-info
  $\Rightarrow A(X)$ and X are "close" to independent.

- The β-approximate max-info between $A(X)$ and X i

$$I_\infty^\beta(A(X);X) = \log\left(\sup_{O} \frac{\Pr[(A(X),X) \in O] - \beta}{\Pr[(A(X'),X) \in O]}\right)$$

Real World

Ideal World

Max-Information of Algorithms [DFHPRR'15]

$$I_\infty^\beta(A(X);X) = \log\left(\sup_O \frac{\Pr[(A(X),X)\in O]-\beta}{\Pr[(A(X'),X)\ \in O]}\right)$$

An algorithm A has β-approximate max-info for data sets of size n if

any data distribution

$$I_\infty^\beta(A;n) = \sup_{\mathcal{S}:X\sim\mathcal{S}}\left\{I_\infty^\beta(A(X);X)\right\}$$

restrict to product distribution

$$I_{\infty,\Pi}^\beta(A;n) = \sup_{P:X\sim P^n}\left\{I_\infty^\beta(A(X);X)\right\}$$

Post-selection Hypothesis Testing

- [RZ'16]: When mutual info $I(X;A(X))$ is bounded, we can control Pr[False Discovery] for adaptively selected tests.

Bounded
Mutual Info
$I(X;A(X))$

$\Longrightarrow$

Bound on
Pr[False Discovery]
when $t \leftarrow A(X)$

Post-selection Hypothesis Testing

- [RZ'16]: When mutual info $I\big(X; A(X)\big)$ is bounded, we can control Pr[False Discovery] for adaptively selected tests.
- [This Paper]: We get a tighter connection via max-info

Bounded Mutual Info $I\big(X; A(X)\big)$ $\Longrightarrow$ Bounded Max-Info $I_\infty^\beta\big(X; A(X)\big)$ $\Longrightarrow$ Bound on Pr[False Discovery] when $t \leftarrow A(X)$

What procedures A have bounded max-info?

- [DFHPRR'15] Max-information bounds for:
  - (Pure) Differential Privacy – algorithmic stability condition.
  - Description Length – log(image size of A)

Differential Privacy [Dwork,McSherry,Nissim,Smith'06]

- A randomized algorithm $A: D^n \to Y$ is (ε, δ)-differentially private if for any neighboring data sets $x, x' \in D^n$ and for any outcome $S \subseteq Y$ we have

$$P(A(x) \in S) \leq e^{\varepsilon} P(A(x') \in S) + \delta$$

If $\delta = 0$ we say **pure** DP, and otherwise **approximate** DP.

Technical Contributions

- [DFHPRR'15] : If $A: D^n \rightarrow T$ is $(\epsilon, 0)$-DP, then for $\beta > 0$,

$$I^{\beta}_{\infty,\Pi}(A; n) \leq \tilde{O}(\epsilon^2 n)$$

$$I^{0}_{\infty}(A; n) \leq O(\epsilon n)$$

- [This paper]: If $A: D^n \rightarrow T$ is (ϵ, δ)-DP, then

$$I^{\beta}_{\infty,\Pi}(A; n) \leq \tilde{O}(\epsilon^2 n) \text{ where } \beta \approx n\sqrt{\frac{\delta}{\epsilon}}$$

- [This paper] (based on [De'12]) : There exists an (ϵ, δ)-DP procedure A where,

$$I^{\beta}_{\infty}(A; n) \approx n \text{ for any } \beta < \frac{1}{2} - \delta$$

Consequences of Positive Result

Theorem: If $A: D^n \to T$ is (ϵ, δ)-DP, then

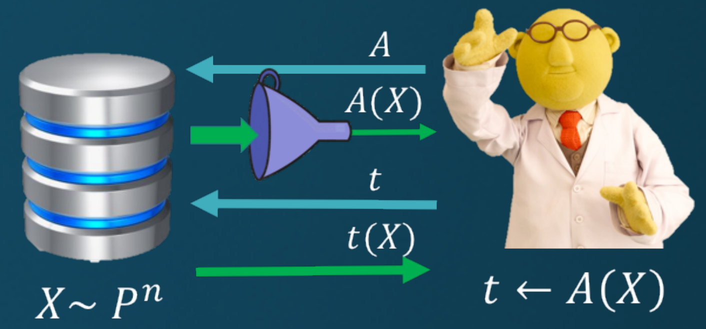
$$I^{\beta}_{\infty,\Pi}(A; n) \leq \tilde{O}(\epsilon^2 n) \text{ where } \beta \approx n\sqrt{\frac{\delta}{\epsilon}}$$

- Recover (optimal) results of [BNSSSU'16] for low sensitive queries.
  - However, our bounds apply more generally (e.g. adaptive hypothesis tests).

- Composition of k adaptively selected $(\epsilon, 0)$-DP procedures: $A_1, \ldots, A_k$
  - [DFHPRR'15]: $I^{\beta}_{\infty,\Pi}(A_k \circ \cdots \circ A_1; n) \leq \tilde{O}(n\epsilon^2 k^2)$
  - [This Paper]: $I^{\beta}_{\infty,\Pi}(A_k \circ \cdots \circ A_1; n) \leq \tilde{O}(n\epsilon^2 k)$

Via strong composition theorem from [Dwork,Rothblum,Vadhan'10]

Contributions



- Post-selection Hypothesis Testing
  - Max-Info Bound $\implies$ bound Pr[False Discovery] in adaptive settings
  - Improves on previous result of [RZ'16] that uses mutual info.
- (ϵ, δ)-DP $\implies$ Bounded Max–Info over product distributions
  - Recovers results from [BNSSSU'16] that dealt with specific analyses.
  - k rounds of adaptivity: we get max-info$\sim k$, where [DFHPRR'15] gives $\sim k^2$

Thanks!

Proof Sketch of Positive Result

Theorem: If $A: D^n \rightarrow T$ is (ϵ, δ)-DP, then

$$I^{\beta}_{\infty,\Pi}(A; n) \leq \tilde{O}(\epsilon^2 n) \text{ where } \beta \approx n\sqrt{\frac{\delta}{\epsilon}}$$

- Define the following random variable where $x \sim P^n, a \sim A(x)$ and

$$Z(a,x) = \log\left(\frac{\Pr[(A(X), X) = (a,x)]}{\Pr[(A(X'), X) = (a,x)]}\right)$$

$$= \sum_{i=1}^{n} \log\left(\frac{\Pr[X_i = x_i \mid a, x_{1:i-1}]}{\Pr[X_i = x_i]}\right) = \sum_{i=1}^{n} Z_i(a, x_{1:i})$$

- Note that if we can bound this with high probability then we can bound approximate max-info.

Proof Sketch of Positive Result

- We want to apply a concentration bound (Azuma's inequality) to the following quantity: $\sum_{i=1}^{n} Z_i(a, x_{1:i})$
- We must then have:
  - A bound on the expectation of each $Z_i(a, x_{1:i})$
  - A bound on each $Z_i(a, x_{1:i})$
- Problem: Each $Z_i(a, x_{1:i})$ is NOT bounded.
- Although each term is bounded with high probability, conditioning on the same $A(X) = a$ and a prefix of data $X_{1:i-1} = x_{1:i-1}$ in every term complicates the argument.

Proof Sketch of Positive Result

For any $t > 0$

$$\Pr\left[\sum_{i=1}^{n} Z_i(A, X_{1:i}) \geq \epsilon^2 n + n\sqrt{\delta/\epsilon} + t\,\epsilon\sqrt{n}\right]$$

$$\leq \Pr\left[\sum_{i=1}^{n} Z_i(A, X_{1:i}) \geq \epsilon^2 n + n\sqrt{\delta/\epsilon} + t\,\epsilon\sqrt{n} \cap (A, X) \in GOOD\right]$$

$$+ \Pr[(A, X) \in BAD]$$

$$\leq e^{\frac{-t^2}{2}} + O\left(n\sqrt{\delta/\epsilon}\right)$$

Set $t = O(\epsilon\sqrt{n})$