

MODULI SPACES OF HYPERBOLIC 3-MANIFOLDS

RICHARD D. CANARY AND PETER A. STORM

ABSTRACT. We study the topology of the moduli space $AI(M)$ of unmarked hyperbolic 3-manifolds homotopy equivalent to a fixed compact hyperbolizable 3-manifold M . This moduli space is the quotient of the more commonly studied space $AH(M)$ of marked hyperbolic 3-manifolds homotopy equivalent to M by the action of $Out(\pi_1(M))$. The deformation space $AH(M)$ is contained in the character variety $X(M)$ associated to M and we also study the dynamics of the action of $Out(\pi_1(M))$ on both $AH(M)$ and on $X(M)$.

For a compact hyperbolizable 3-manifold M , the deformation space $AH(M)$ of marked hyperbolic 3-manifolds homotopy equivalent to M is a familiar object of study. This deformation space sits naturally inside the character variety $X(M)$ and the outer automorphism group $Out(\pi_1(M))$ acts by homeomorphisms on both $AH(M)$ and $X(M)$. The action of $Out(\pi_1(M))$ on $AH(M)$ and $X(M)$ has largely been studied in the case when M is an interval bundle over a closed surface (see, for example, [8, 21, 45, 17]) or a handlebody (see, for example, [41, 50]). In this paper, we initiate a study of this action and its quotient for general hyperbolizable 3-manifolds.

We begin by studying the topological quotient

$$AI(M) = AH(M)/Out(\pi_1(M))$$

which we may think of as the moduli space of unmarked hyperbolic 3-manifolds homotopy equivalent to M . The space $AH(M)$ is a rather pathological topological object itself, often failing to even be locally connected (see Bromberg [11] and Magid [33]). However, since $AH(M)$ is a closed subset of an open submanifold of the character variety, it does retain many nice topological properties. We will see that the topology of $AI(M)$ reflects the topology of M .

The first hint that $AI(M)$ may be even worse topologically than $AH(M)$, is Thurston's proof [47] that if M is homeomorphic to $S \times I$, then there are infinite order elements of $Out(\pi_1(M))$ which have fixed points in $AH(M)$. (These elements are pseudo-Anosov mapping classes in $Mod_+(S) \subset Out(\pi_1(S))$.)

Date: November 9, 2009.

The first author was partially supported by National Science Foundation grants DMS-0504791 and DMS-0554239. The second author is partially supported by National Science Foundation grant DMS-0904355.

One may further show that $AI(S \times I)$ is not even T_1 , see [17] for a closely related result. Recall that a topological space is T_1 if all its points are closed. On the other hand, we show that in all other cases $AI(M)$ is T_1 .

Theorem 1.1. *Let M be a compact hyperbolizable 3-manifold with non-abelian fundamental group. Then the moduli space $AI(M)$ is T_1 if and only if M is not an untwisted interval bundle.*

We next turn to the question of determining when $AI(M)$ is Hausdorff. If M contains a primitive essential annulus, then we use the Hyperbolic Dehn Filling Theorem to show that $AI(M)$ is not Hausdorff. A properly embedded annulus in M is a primitive essential annulus if it cannot be properly isotoped into the boundary of M and its core curve generates a maximal abelian subgroup of $\pi_1(M)$. In particular, if M has compressible boundary and no toroidal boundary components, then $AI(M)$ is not Hausdorff (see Corollary 7.4).

Theorem 1.2. *Let M be a compact hyperbolizable 3-manifold with non-abelian fundamental group. If M contains a primitive essential annulus then $AI(M)$ is not Hausdorff. In particular, if M contains a primitive essential annulus, then $Out(\pi_1(M))$ does not act properly discontinuously on $AH(M)$.*

On the other hand, if M is acylindrical, i.e. has incompressible boundary and contains no essential annuli, then $Out(\pi_1(M))$ is finite (see Johannson [28, Proposition 27.1]), so $Out(\pi_1(M))$ acts properly discontinuously on $AH(M)$ and $X(M)$, so $AI(M)$ is Hausdorff.

If M is a compact hyperbolizable 3-manifold which is not acylindrical, but does not contain any primitive essential annuli, then $Out(\pi_1(M))$ is infinite. However, if, in addition, M has no toroidal boundary components, we show that $Out(\pi_1(M))$ acts properly discontinuously on an open neighborhood of $AH(M)$ in $X(M)$. In particular, we see that $AI(M)$ is Hausdorff in this case.

Theorem 1.3. *If M is a compact hyperbolizable 3-manifold with no primitive essential annuli whose boundary has no toroidal boundary components, then there exists an open $Out(\pi_1(M))$ -invariant neighborhood $W(M)$ of $AH(M)$ in $X(M)$ such that $Out(\pi_1(M))$ acts properly discontinuously on $W(M)$. In particular, $AI(M)$ is Hausdorff.*

If M is a compact hyperbolizable 3-manifold with no primitive essential annuli whose boundary has no toroidal boundary components, then $Out(\pi_1(M))$ is virtually abelian (see the discussion in sections 5 and 9). However, we note that the conclusion of Theorem 1.3 relies crucially on the topology of M , not just the group theory of $Out(\pi_1(M))$. In particular, if M is a compact hyperbolizable 3-manifold M with incompressible boundary, such that every component of its characteristic submanifold is a solid torus, then $Out(\pi_1(M))$ is always virtually abelian, but M may contain primitive essential annuli, in which case $Out(\pi_1(M))$ does not act properly discontinuously on $AH(M)$.

One may combine Theorem 1.2 and 1.3 to completely characterize when $AI(M)$ is Hausdorff when M has no toroidal boundary components.

Corollary 1.4. *Let M be a compact hyperbolizable 3-manifold with no toroidal boundary components and non-abelian fundamental group. The moduli space $AI(M)$ is Hausdorff if and only if M contains no primitive essential annuli.*

It is a consequence of the classical deformation theory of Kleinian groups (see Bers [5] or Canary-McCullough [16, Chapter 7] for a survey of this theory) that $Out(\pi_1(M))$ acts properly discontinuously on the interior $\text{int}(AH(M))$ of $AH(M)$. If H_n is the handlebody of genus $n \geq 2$, Minsky [41] exhibited an explicit $Out(\pi_1(H_n))$ -invariant open subset $PS(H_n)$ of $X(H_n)$ such that $\text{int}(AH(H_n))$ is a proper subset of $PS(H_n)$ and $Out(\pi_1(H_n))$ acts properly discontinuously on $AH(H_n)$.

If M is a compact hyperbolizable 3-manifold with incompressible boundary and no toroidal boundary components, which is not an interval bundle, then we find an open set strictly bigger than $\text{int}(AH(M))$ which $Out(\pi_1(M))$ acts properly discontinuously on. See Theorem 9.1 and its proof for a more precise description of $W(M)$.

Theorem 1.5. *Let M be a compact hyperbolizable 3-manifold with nonempty incompressible boundary and no toroidal boundary components, which is not an interval bundle. Then there exists an open $Out(\pi_1(M))$ -invariant subset $W(M)$ of $X(M)$ such that $Out(\pi_1(M))$ acts properly discontinuously on $W(M)$ and $\text{int}(AH(M))$ is a proper subset of $W(M)$.*

It is conjectured that if M is an untwisted interval bundle over a closed surface S , then $\text{int}(AH(M))$ is the maximal open $Out(\pi_1(M))$ -invariant subset of $X(M)$ on which $Out(\pi_1(M))$ acts properly discontinuously. Evidence for this conjecture is provided by results of Bowditch [8], Goldman [20], Souto-Storm [45], Tan-Wong-Zhang [50] and Cantat [18]. For example, methods of Souto and Storm [45] show that if W is an $Out(\pi_1(S))$ -invariant open subset of $X(S \times I)$ which $Out(\pi_1(S))$ acts properly discontinuously on, then W cannot intersect $\partial AH(S \times I)$.

Michelle Lee [32] has recently shown that if M is a twisted interval bundle over a closed surface, then there exists an open $Out(\pi_1(M))$ -invariant subset W of $X(M)$ such that $Out(\pi_1(M))$ acts properly discontinuously on W and $\text{int}(AH(M))$ is a proper subset of W . Moreover, W contains points in $\partial AH(M)$. Combining this result, with our work and Souto-Storm [45], she proves that if M has incompressible boundary and no toroidal boundary components, then there is open $Out(\pi_1(M))$ -invariant subset W of $X(M)$ such that $Out(\pi_1(M))$ acts properly discontinuously on W , $\text{int}(AH(M))$ is a proper subset of W , and $W \cap \partial AH(M) \neq \emptyset$ if and only if M is not an untwisted interval bundle.

Outline of paper: In section 2, we recall background material from topology and hyperbolic geometry which will be used in the paper.

In section 3, we prove Theorem 1.1. The proof that $AI(S \times I)$ is not T_1 follows the arguments in [17, Proposition 3.1] closely. We now sketch the proof that $AI(M)$ is T_1 otherwise. In this case, let $N \in AI(M)$ and let R be a compact core for N . We show that N is a closed point, by showing that any convergent sequence $\{\rho_n\}$ in the pre-image of N is eventually constant. For all n , there exists a homotopy equivalence $h_n : M \rightarrow N$ such that $(h_n)_* = \rho_n$. If G is a graph in M carrying $\pi_1(M)$, then, since $\{\rho_n\}$ is convergent, we can assume that the length of $h_n(G)$ is at most K , for all n and some K . But, we observe that $h_n(G)$ cannot lie entirely in the complement of R , if R is not a compression body. In this case, each $h_n(G)$ lies in the compact neighborhood of radius K of R , so there are only finitely many possible homotopy classes of maps of G . Thus, there are only finitely many possibilities for ρ_n , so $\{\rho_n\}$ is eventually constant. The proof in the case that R is a compression body is somewhat more complicated and uses the Covering Theorem.

In section 4, we prove Theorem 1.2. Let A be a primitive essential annulus in M . If α is a core curve of A , then the complement \hat{M} of a regular neighborhood of α in M is hyperbolizable. We consider a geometrically finite hyperbolic manifold \hat{N} homeomorphic to the interior of \hat{M} which has two covers N_0 and N_1 which are homeomorphic to the interior of M but are not isometric. We then use the Hyperbolic Dehn Filling Theorem to produce sequences $\{\rho_n\}$ in $AH(M)$ and $\varphi_n \in Out(\pi_1(M))$ such that $\{\rho_n\}$ converges to ρ_0 with $N_{\rho_0} = N_0$ and $\{\rho_n \circ \varphi_n\}$ converges to ρ_1 with $N_{\rho_1} = N_1$. Therefore, $\{\rho_n\}$ projects to a sequence in $AI(M)$ with two distinct limits, so $AI(M)$ is not Hausdorff.

In section 5 we recall basic facts about the characteristic submanifold and the mapping class group of compact hyperbolizable 3-manifolds with incompressible boundary and no toroidal boundary components. We identify a finite index subgroup of $Out(\pi_1(M))$ which is the direct product of the free abelian subgroup generated by Dehn twists in frontier annuli of the characteristic submanifold and mapping class groups of base surfaces of interval bundles in the characteristic submanifold.

In section 6, we organize the frontier annuli of the characteristic submanifold into characteristic collections of annuli and describe free subgroups of $\pi_1(M)$ which register the action of the subgroup of $Out(\pi_1(M))$ generated by Dehn twists in the annuli in such a collection.

In section 7, we show that compact hyperbolizable 3-manifolds with compressible boundary and no toroidal boundary components contain primitive essential annuli.

In section 8, we introduce the space $AH_n(M)$ of discrete faithful representation such that $\rho(\pi_1(\Sigma_i))$ is purely hyperbolic for every component Σ_i of the characteristic submanifold of M (where we continue to assume that M has incompressible boundary and no toroidal boundary components). We see that $\text{int}(AH(M))$ is a proper subset of $AH_n(M)$ and $AH_n(M) = AH(M)$ if M does not contain any primitive essential annuli.

In section 9, we prove that if M has incompressible boundary and no toroidal boundary components, but is not an interval bundle, there is an open neighborhood $W(M)$ of $AH_n(M)$ in $X(M)$ such that $Out(\pi_1(M))$ preserves and acts properly discontinuously on $W(M)$. Theorems 1.3 and 1.5 are immediate corollaries. We finish the outline by sketching the proof in a special case.

Let X be an acylindrical, compact hyperbolizable 3-manifold and let A be an incompressible annulus in its boundary. Let V be a solid torus and let B_1, \dots, B_n be a collection of disjoint parallel annuli in ∂V whose core curves are homotopic to the n^{th} power of the core curve of V where $|n| \geq 2$. Let M_1, \dots, M_n be copies of X and let A_1, \dots, A_n be copies of A in M_i . We form M by attaching each M_i to V by identifying A_i and B_i . Then M contains no primitive essential annuli, is hyperbolizable, and $Out(\pi_1(M))$ has a finite index subgroup $J(M)$ generated by Dehn twists about A_1, \dots, A_n . In particular, $J(M) \cong \mathbf{Z}^{n-1}$.

In this case, $\{A_1, \dots, A_n\}$ is the only characteristic collection of annuli. We say that a group H registers $J(M)$ if it is freely generated by the core curve of V and, for each i , a curve contained in $V \cup M_i$ which is not homotopic into V . So $H \cong F_{n+1}$. There is a natural map $r_H : X(M) \rightarrow X(H)$ where $X(H)$ is the $PSL_2(\mathbf{C})$ -character variety of the group H . Notice that $J(M)$ preserves H and injects into $Out(H)$. Let

$$\mathcal{S}_{n+1} = \text{int}(AH(H)) \subset X(H)$$

denote the space of Schottky representations (i.e. representations which are purely hyperbolic and geometrically finite.) Since $Out(H)$ acts properly discontinuously on \mathcal{S}_{n+1} , we see that $J(M)$ acts properly discontinuously on

$$W_H = r_H^{-1}(\mathcal{S}_{n+1})$$

Let $W = \bigcup W_H$ where the union is taken over all subgroups which register $J(M)$. Notice that W is open and $J(M)$ acts properly discontinuously on W . One may use a ping pong argument to show that $AH_n(M) \subset W$, see Lemma 8.3. Johannson's Classification Theorem is used to show that W is invariant under $Out(\pi_1(M))$, see Lemma 9.6. (Actually, we define a somewhat larger set, in general, by using the space of primitive-stable representations in place of Schottky space.)

Acknowledgements: Both authors would like to thank Yair Minsky for helpful and interesting conversations. The second author thanks the organizers of the workshop "Dynamics of $Aut(F_n)$ -actions on representation varieties," held in Midreshet Sde Boker, Israel in January, 2009. We thank Michelle Lee for helpful comments on a preliminary version of this paper.

2. PRELIMINARIES

As a convention, the letter M will denote a compact connected oriented hyperbolizable 3-manifold. We recall that M is said to be hyperbolizable if

the interior of M admits a complete hyperbolic metric. We will use N to denote a hyperbolic 3-manifold. All hyperbolic 3-manifolds are assumed to be oriented, complete, and connected.

2.1. The deformation spaces. Recall that $\mathrm{PSL}_2(\mathbf{C})$ is the group of orientation-preserving isometries of \mathbf{H}^3 . Given a 3-manifold M , a discrete, faithful representation $\rho : \pi_1(M) \rightarrow \mathrm{PSL}_2(\mathbf{C})$ determines a hyperbolic 3-manifold $N_\rho = \mathbf{H}^3/\rho(\pi_1(M))$ and a homotopy equivalence $m_\rho : M \rightarrow N_\rho$, called the marking of N_ρ .

We let $D(M)$ denote the set of discrete, faithful representations of $\pi_1(M)$ into $\mathrm{PSL}_2(\mathbf{C})$. The group $\mathrm{PSL}_2(\mathbf{C})$ acts by conjugation on $D(M)$ and we let

$$AH(M) = D(M)/\mathrm{PSL}_2(\mathbf{C}).$$

Elements of $AH(M)$ are hyperbolic 3-manifolds homotopy equivalent to M equipped with (homotopy classes of) markings.

The space $AH(M)$ is a subset of the character variety

$$X(M) = \mathrm{Hom}_T(\pi_1(M), \mathrm{PSL}_2(\mathbf{C}))//\mathrm{PSL}_2(\mathbf{C}),$$

which is the Mumford quotient of the space $\mathrm{Hom}_T(\pi_1(M), \mathrm{PSL}_2(\mathbf{C}))$ of representations $\rho : \pi_1(M) \rightarrow \mathrm{PSL}_2(\mathbf{C})$ such that $\rho(g)$ is parabolic if $g \neq id$ lies in a rank two free abelian subgroup of $\pi_1(M)$. If M has no toroidal boundary components, then $\mathrm{Hom}_T(\pi_1(M), \mathrm{PSL}_2(\mathbf{C}))$ is simply $\mathrm{Hom}_T(\pi_1(M), \mathrm{PSL}_2(\mathbf{C}))$. Moreover, each point in $AH(M)$ is a smooth point of $X(M)$ (see Kapovich [29, Sections 4.3 and 8.8] and Heusener-Porti [23] for more details on this construction).

The group $\mathrm{Aut}(\pi_1(M))$ acts naturally on $\mathrm{Hom}_T(\pi_1(M), \mathrm{PSL}_2(\mathbf{C}))$ via

$$(\varphi \cdot \rho)(\gamma) := \rho(\varphi^{-1}(\gamma)).$$

This descends to an action of $\mathrm{Out}(\pi_1(M))$ on $AH(M)$ and $X(M)$. This action is not free, and it often has complex dynamics. Nonetheless, we can define the topological quotient space

$$AI(M) = AH(M)/\mathrm{Out}(\pi_1(M)).$$

Elements of $AI(M)$ are naturally hyperbolic 3-manifolds homotopy equivalent to M without a specified marking.

2.2. Topological background. A compact 3-manifold M is said to have *incompressible boundary* if whenever S is a component of ∂M , the inclusion map induces an injection of $\pi_1(S)$ into $\pi_1(M)$. In our setting, this is equivalent to $\pi_1(M)$ being freely indecomposable. A properly embedded annulus A in M is said to be *essential* if the inclusion map induces an injection of $\pi_1(A)$ into $\pi_1(M)$ and A cannot be properly homotoped into ∂M (i.e. there does not exist a homotopy of pairs of the inclusion $(A, \partial A) \rightarrow (M, \partial M)$ to a map with image in ∂M). An essential annulus A is said to be *primitive* if the image of $\pi_1(A)$ in $\pi_1(M)$ is a maximal abelian subgroup.

If M does not have incompressible boundary, it is said to have *compressible boundary*. The fundamental examples of 3-manifolds with compressible boundary are compression bodies. Let S be a closed, possibly disconnected, orientable surface without boundary, and consider the 3-manifold $S \times I$. A *compression body* is formed by attaching 1-handles to disjoint disks on the boundary surface $S \times \{1\}$. The resulting 3-manifold C (assumed to be connected) will have a single boundary component $\partial_+ C$ intersecting $S \times \{1\}$, called the positive (or external) boundary of C . If C is not an untwisted interval bundle over a closed surface, then $\partial_+ C$ is the unique compressible boundary component of C . Notice that the induced homomorphism $\pi_1(\partial_+ C) \rightarrow \pi_1(C)$ is surjective. In fact, a compact irreducible 3-manifold M is a compression body if and only if there exists a component S of ∂M such that $\pi_1(S) \rightarrow \pi_1(M)$ is surjective.

Every compact hyperbolizable 3-manifold can be constructed from compression bodies and manifolds with incompressible boundary. Bonahon [6] and McCullough-Miller [38] showed that there exists a neighborhood C_M of ∂M , called the *characteristic compression body*, such that each component of C_M is a compression body and each component of $\partial C_M - \partial M$ is incompressible in M .

Dehn filling will play a key role in the proof of Theorem 1.2. Let F be a toroidal boundary component of compact 3-manifold M and let (m, l) be a choice of meridian and longitude for F . Given a pair (p, q) of relatively prime integers, we may form a new manifold $M(p, q)$ by attaching a solid torus V to M by an orientation-reversing homeomorphism $g: \partial V \rightarrow F$ so that, if c is the meridian of V , then $g(c)$ is a (p, q) curve on F with respect to the chosen meridian-longitude system. We say that $M(p, q)$ is obtained from M by (p, q) -Dehn filling along F .

2.3. Hyperbolic background. If $N = \mathbb{H}^3/\Gamma$ is a hyperbolic 3-manifold, then $\Gamma \subset \mathrm{PSL}_2(\mathbb{C})$ acts on $\widehat{\mathbb{C}}$ as a group of conformal automorphisms. The *domain of discontinuity* $\Omega(\Gamma)$ is the largest open Γ -invariant subset of $\widehat{\mathbb{C}}$ on which Γ acts properly discontinuously. Note that $\Omega(\Gamma)$ may be empty. Its complement $\Lambda(\Gamma) = \widehat{\mathbb{C}} - \Omega(\Gamma)$ is called the *limit set*. The quotient $\partial_c N = \Omega(\Gamma)/\Gamma$ is naturally a Riemann surface called the *conformal boundary*.

Thurston's Hyperbolization theorem, see Morgan [42, Theorem B'], guarantees that if M is compact and hyperbolizable, then there exists a hyperbolic 3-manifold N and a homeomorphism

$$\psi : M - \partial_T M \rightarrow N \cup \partial_c N$$

where $\partial_T M$ denotes the collection of toroidal boundary components of M .

The convex core $C(N)$ of N is the smallest convex submanifold whose inclusion into N is a homotopy equivalence. More concretely, it is obtained as the quotient, by Γ , of the convex hull, in \mathbb{H}^3 , of the limit set $\Lambda(\Gamma)$. There is a well-defined retraction $r : N \rightarrow C(N)$ obtained by taking x to the (unique) point in $C(N)$ closest to x . The nearest point retraction r is a homotopy

equivalence and is $\frac{1}{\cosh s}$ -Lipschitz on the complement of the neighborhood of radius s of $C(N)$.

There exists a universal constant μ , called the Margulis constant, such that if $\epsilon < \mu$, then each component of the ϵ -thin part

$$N_{thin(\epsilon)} = \{x \in N \mid \text{inj}_N(x) < \epsilon\}$$

(where $\text{inj}_N(x)$ denotes the injectivity radius of N at x) is either a metric regular neighborhood of a geodesic or is homeomorphic to $T \times (0, \infty)$ where T is either a torus or an open annulus (see Benedetti-Petronio [4] for example). The ϵ -thick part of N is defined simply to be the complement of the ϵ -thin part

$$N_{thick(\epsilon)} = N - N_{thin(\epsilon)}.$$

It is also useful to consider the manifold N_ϵ^0 obtained from N by removing the non-compact components of $N_{thin(\epsilon)}$.

If N is a hyperbolic 3-manifold with finitely generated fundamental group, then it admits a compact core, i.e. a compact submanifold whose inclusion into M is a homotopy equivalence (see Scott [44]). More generally, if $\epsilon < \mu$, then there exists a *relative compact core* R for N_ϵ^0 , i.e. a compact core which intersects each component of ∂N_ϵ^0 in a compact core for that component (see Kulkarni-Shalen [31] or McCullough [36]). Let $P = \partial R - \partial N_\epsilon^0$ and let P^0 denote the interior of P . The Tameness Theorem of Agol [1] and Calegari-Gabai [13] assures us that we may choose R so that $N_\epsilon^0 - R$ is homeomorphic to $(\partial R - P^0) \times (0, \infty)$. In particular, the ends of N_ϵ^0 are in one-to-one correspondence with the components of $\partial R - P^0$. (We will blur this distinction and simply regard an end as a component of $N_\epsilon^0 - R$ once we have chosen ϵ and a relative compact core R for N_ϵ^0 .) We say that an end U of N_ϵ^0 is *geometrically finite* if the intersection of $C(N)$ with U is bounded (i.e. admits a compact closure). N is said to be *geometrically finite* if all the ends of N_ϵ^0 are geometrically finite.

Thurston [49] showed that if M is a compact hyperbolizable 3-manifold whose boundary is a torus F , then all but finitely many Dehn fillings of M are hyperbolizable. Moreover, as the Dehn surgery coefficients approach ∞ , the resulting hyperbolic manifolds “converge” to the hyperbolic 3-manifold homeomorphic to $\text{int}(M)$. If M has other boundary components, then there is a version of this theorem where one begins with a geometrically finite hyperbolic 3-manifold homeomorphic to $\text{int}(M)$ and one is allowed to perform the Dehn filling while fixing the conformal structure on the non-toroidal boundary components of M . The proof uses the cone-manifold deformation theory developed by Hodgson-Kerckhoff [24] in the finite volume case and extended to the infinite volume case by Bromberg [10] and Brock-Bromberg [9]. (The first statement of a Hyperbolic Dehn Filling Theorem in the infinite volume setting was given by Bonahon-Otal [7], see also Comar [19].) For a statement of the Filling Theorem in this form, and a discussion of its derivation from the previously mentioned work, see Bromberg [11] or Magid [33].

Hyperbolic Dehn Filling Theorem: *Let M be a compact, hyperbolizable 3-manifold and let F be a toroidal boundary component of M . Let $N = \mathbb{H}^3/\Gamma$ be a hyperbolic 3-manifold admitting an orientation-preserving homeomorphism $\psi : M - \partial_T M \rightarrow N \cup \partial_c N$. Let $\{(p_n, q_n)\}$ be an infinite sequence of distinct pairs of relatively prime integers.*

Then, for all sufficiently large n , there exists a (non-faithful) representation $\beta_n : \Gamma \rightarrow \mathrm{PSL}_2(\mathbf{C})$ with discrete image such that

- (1) $\{\beta_n\}$ converges to the identity representation of Γ , and
- (2) if $i_n : M \rightarrow M(p_n, q_n)$ denotes the inclusion map, then for each n , there exists an orientation-preserving homeomorphism

$$\psi_n : M(p_n, q_n) - \partial_T M(p_n, q_n) \rightarrow N_{\beta_n} \cup \partial_c N_{\beta_n}$$

such that $\beta_n \circ \psi_$ is conjugate to $(\psi_n)_* \circ (i_n)_*$, and the restriction of $\psi_n \circ i_n \circ \psi^{-1}$ to $\partial_c N$ is conformal.*

3. POINTS ARE USUALLY CLOSED

If S is a closed orientable surface, we showed in [17] that $\mathcal{AI}(S) = AH(S \times I)/\mathrm{Mod}_+(S)$ is not T_1 where $\mathrm{Mod}_+(S)$ is the group of (isotopy classes of) orientation-preserving homeomorphisms of S . We recall that a topological space is T_1 if all points are closed sets. Since $\mathrm{Mod}_+(S)$ is identified with an index two subgroup of $\mathrm{Out}(\pi_1(S))$, one also expects that $AI(S \times I) = AH(S \times I)/\mathrm{Out}(\pi_1(S))$ is not T_1 .

In this section, we show that if M is an untwisted interval bundle, which also includes the case that M is a handlebody, then $AI(M)$ is not T_1 , but that $AI(M)$ is T_1 for all other compact, hyperbolizable 3-manifolds.

Theorem 1.1. *Let M be a compact hyperbolizable 3-manifold with non-abelian fundamental group. Then the moduli space $AI(M)$ is T_1 if and only if M is not an untwisted interval bundle.*

Proof. We first show that $AI(M)$ is T_1 if M is not an untwisted interval bundle. Let $p : AH(M) \rightarrow AI(M)$ be the quotient map and let N be a manifold in $AI(M)$. We must show that $p^{-1}(N)$ is a closed subset of $AH(M)$. Since $AH(M)$ is Hausdorff and second countable, it suffices to show that if $\{\rho_n\}$ is a convergent sequence in $p^{-1}(N)$, then $\lim \rho_n \in p^{-1}(N)$.

An element $\rho \in p^{-1}(N)$ is a representation such that N_ρ is isometric to N . Let $\{\rho_n\}$ be a convergent sequence of representations in $p^{-1}(N)$. Let $G \subset M$ be a finite graph such that the inclusion map induces a surjection of $\pi_1(G)$ onto $\pi_1(M)$. Each ρ_n gives rise to a homotopy equivalence $h_n : M \rightarrow N$, and hence to a map $j_n = h_n|_G : G \rightarrow N$, both of which are only well-defined up to homotopy. Since $\{\rho_n\}$ is convergent, there exists K such that $j_n(G)$ has length at most K for all n , after possibly altering h_n by a homotopy.

Let R be a compact core for N . Assume first that R is not a compression body. In this case, if S is any component of ∂R , then the inclusion map does not induce a surjection of $\pi_1(S)$ to $\pi_1(R)$ (see the discussion in section

2). Since $j_n(G)$ carries the fundamental group it cannot lie entirely outside of R . It follows that $j_n(G)$ lies in the closed neighborhood $\mathcal{N}_K(R)$ of radius K about R . By compactness, there are only finitely many homotopy classes of maps of G into $\mathcal{N}_K(R)$ with total length at most K . Hence, there are only finitely many different representations among the ρ_n , up to conjugacy. The deformation space $AH(M)$ is Hausdorff, and the sequence $\{\rho_n\}$ converges, implying that $\{\rho_n\}$ is eventually constant. Therefore $\lim \rho_n$ lies in the preimage of N , implying that the fiber $p^{-1}(N)$ is closed and that N is a closed point of $AI(M)$.

Next we assume that R is a compression body. If R were an untwisted interval bundle, then M would also have to be a untwisted interval bundle (by Theorems 5.2 and 10.6 in Hempel [22]) which we have disallowed. So R must have at least one incompressible boundary component and only one compressible boundary component ∂_+R . We are free to assume that M is homeomorphic to R , since the definition of $AI(M)$ depends only on the homotopy type of M . Let D denote the union of R and the component of $N - R$ bounded by ∂_+R . Since the fundamental group of a component of $N - D$ never surjects onto $\pi_1(N)$, with respect to the map induced by inclusion, we see as above that each $j_n(G)$ must intersect D , so is contained in the neighborhood of radius K of D .

Recall that there exists $\epsilon_K > 0$ so that the distance from the ϵ_K -thin part of N to the μ -thick part of N is greater than K (where μ is the Margulis constant). It follows that $j_n(G)$ must be contained in the ϵ_K -thick part of N .

Let F be an incompressible boundary component of M . Then $h_n(F)$ is homotopic to an incompressible boundary component of R (see, for example, the proof of Proposition 9.2.1 in [16]). As there are finitely many possibilities, we may pass to a subsequence so that $h_n(F)$ is homotopic to a fixed boundary component F' . We may choose G so that there is a proper subgraph $G_F \subset G$ such that the image of $\pi_1(G_F)$ in $\pi_1(M)$ (under the inclusion map) is conjugate to $\pi_1(F)$. Let $p_F : N_F \rightarrow N$ be the covering map associated to $\pi_1(F') \subset \pi_1(N)$. Then $j_n|_{G_F}$ lifts to a map k_n of G_F into N_F .

Assume first that F is a torus. Then $k_n(G_F)$ must lie in the portion X of N_F with injectivity radius between ϵ_K and $K/2$, which is compact. It follows that $j_n(G)$ must lie in the closed neighborhood of radius K of $p_F(X)$. Since $p_F(X)$ is compact, we may conclude, as in the general case, that $\{\rho_n\}$ is eventually constant and hence that $p^{-1}(N)$ is closed.

We now suppose that F has genus at least 2. We first establish that there exists L such that $k_n(G_F)$ must be contained in a neighborhood of radius L of the convex core $C(N_F)$. It is a consequence of the thick-thin decomposition, that if G' is a graph in N_F which carries the fundamental group then G' must have length at least μ . We also recall that the nearest point retraction $r_F : N_F \rightarrow C(N_F)$ is a homotopy equivalence which is $\frac{1}{\cosh s}$ -Lipschitz on the complement of the neighborhood of radius s of $C(N)$. Therefore, if $k_n(G_F)$ lies outside of $\mathcal{N}_s(C(N_F))$, then $r_F(k_n(G_F))$ has length

at most $\frac{K}{\cosh s}$. It follows that $k_n(G_F)$ must intersect the neighborhood of radius $\cosh^{-1}(\frac{K}{\mu})$ of $C(N_F)$, so we may choose $L = K + \cosh^{-1}(\frac{K}{\mu})$.

If N_F is geometrically finite, then $X = C(N_F) \cap N_{thick(\epsilon_K)}$ is compact and $j_n(G)$ must be contained in the neighborhood of radius $L + K$ of $p_F(X)$ which allows us to complete the proof as before.

If N_F is not geometrically finite, we will need to invoke the Covering Theorem to complete the proof. Let \tilde{F} denote the lift of F' to N_F . Then \tilde{F} divides N_F into two components, one of which, say A_- , is mapped homeomorphically to the component of $N - R$ bounded by F' . Let $A_+ = N_F - A_-$. We may choose a relative compact core R_F for $(N_F)_\epsilon^0$ (for some $\epsilon < \epsilon_K$) so that \tilde{F} is contained in the interior of R_F . Since p_F is infinite-to-one on each end of $(N_F)_\epsilon^0$ which is contained in A_+ , the Covering Theorem (see [14] or [49]) implies that all such ends are geometrically finite. Therefore,

$$Y = A_+ \cap C(N_F) \cap (N_F)_{thick(\epsilon_K)}$$

is compact. If we let $Z = A_- \cup Y$, then we see that $k_n(G_F)$ is contained in the closed neighborhood of radius L about Z (since $C(N_F) \cap N_{thick(\epsilon_K)} \subset Z$). Therefore, $j_n(G)$ is contained in the closed $(L + K)$ -neighborhood of $D \cap p_F(Z) = D \cap p_F(Y)$. Since $D \cap p_F(Y)$ is compact, we conclude, exactly as in the previous cases, that $p^{-1}(N)$ is closed. This case completes the proof that $AI(M)$ is T_1 if M is not an untwisted interval bundle.

We now deal with the case where $M = S \times I$ is an untwisted interval bundle over a compact surface S . In our previous paper [17], we consider the space $AH(S)$ of (conjugacy classes of) discrete faithful representations $\rho : \pi_1(S) \rightarrow \mathrm{PSL}_2(\mathbf{C})$ such that if $g \in \pi_1(S)$ is peripheral, then $\rho(g)$ is parabolic. In Proposition 3.1, we use work of Thurston [47] and McMullen [39] to exhibit a sequence $\{\rho_n\}$ in $AH(S)$ which converges to $\rho \in AH(S)$ such that $\Lambda(\rho) = \widehat{\mathbf{C}}$, $\Lambda(\rho_1) \neq \widehat{\mathbf{C}}$ and for all n there exists $\varphi_n \in \mathrm{Mod}(S)$ such that $\rho_n = \rho_1 \circ \varphi_n$. Since $AH(S) \subset AH(S \times I)$ and $\mathrm{Mod}(S)$ is identified with a subgroup of $\mathrm{Out}(\pi_1(S))$, we see that $\{\rho_n\}$ is a sequence in $p^{-1}(N_{\rho_1})$ which converges to a point outside of $p^{-1}(N_{\rho_1})$. Therefore, N_{ρ_1} is a point in $AI(S \times I)$ which is not closed. \square

Remark: One may further show, as in the remark after Proposition 3.1 in [17], that if $N \in AI(S \times I)$ is a degenerate hyperbolic 3-manifold with a lower bound on its injectivity radius, then N is not a closed point in $AI(S \times I)$. We recall that $N = \mathbb{H}^3/\Gamma$ is degenerate if $\Omega(\Gamma)$ is connected and simply connected and Γ is finitely generated.

4. PRIMITIVE ESSENTIAL ANNULI AND THE FAILURE TO BE HAUSDORFF

In this section, we show that $AI(M)$ is not Hausdorff if M contains a primitive essential annulus. We do so by using the Hyperbolic Dehn Filling Theorem to produce two convergent sequences in $AH(M)$ with non-isometric limits which project to the same sequence in $AI(M)$. The construction is a

generalization of a construction of Kerckhoff-Thurston [30]. One may also think of the argument as a simple version of the “wrapping” construction (see Anderson-Canary [2]) which was also used to show that components of $\text{int}(AH(M))$ self-bump whenever M contains a primitive essential annulus (see McMullen [40] and Bromberg-Holt [12]).

Theorem 1.2. *Let M be a compact hyperbolizable 3-manifold with non-abelian fundamental group. If M contains a primitive essential annulus then $AI(M)$ is not Hausdorff.*

Proof. Let A be a primitive essential annulus in M with core curve α . Let $\hat{M} = M - \mathcal{N}(\alpha)$. Lemma 10.2 in [3] observes that \hat{M} is hyperbolizable. Since \hat{M} is hyperbolizable, Thurston’s Hyperbolization Theorem implies that there exists a hyperbolic manifold \hat{N} and a homeomorphism $\psi : \hat{M} - \partial_T \hat{M} \rightarrow \hat{N} \cup \partial_c \hat{N}$. The classical deformation theory of Kleinian groups (see Bers [5] or [16]) implies that we may choose any conformal structure on $\partial_c \hat{N}$.

Let A_0 and A_1 denote the components of $A \cap \hat{M}$. Let M_i be the complement in \hat{M} of a regular neighborhood of A_i . Let $h_i : M \rightarrow \hat{M}$ be an embedding with image M_i which agrees with the identity map off of a (somewhat larger) regular neighborhood of A .

Let F be the toroidal boundary component of \hat{M} which is the boundary of $\mathcal{N}(A)$ in M . Choose a meridian-longitude system for F so that the meridian for F bounds a disk in M . Lemma 10.3 in [3] implies that if $i_n : \hat{M} \rightarrow \hat{M}(1, n)$ is the inclusion map, then $i_n \circ h_i : M \rightarrow \hat{M}(1, n)$ is homotopic to a homeomorphism for each $i = 1, 2$ and all $n \in \mathbb{Z}$. Moreover, we may similarly check that $i_n \circ h_1$ is homotopic to $i_n \circ h_0 \circ D_A^n$ for all n , where D_A denotes a Dehn twist along A . Notice first that $j_n = D_{A_0}^n$ takes a $(1, 0)$ -curve on F to a $(1, n)$ -curve on F , so extends to a homeomorphism $j_n : M = \hat{M}(1, 0) \rightarrow \hat{M}(1, n)$. Therefore, since $i_0 \circ h_0$ and $i_0 \circ h_1$ are homotopic, so are $j_n \circ i_0 \circ h_0$ and $j_n \circ i_0 \circ h_1$. But, $j_n \circ i_0 \circ h_0$ is homotopic to $i_n \circ h_0 \circ D_A^n$ and $j_n \circ i_0 \circ h_1 = i_n \circ h_1$, which completes the proof that $i_n \circ h_1$ is homotopic to $i_n \circ h_0 \circ D_A^n$ for all n .

Let $\rho_0 = (\psi \circ h_0)_*$ and $\rho_1 = (\psi \circ h_1)_*$. Since $(h_i)_*$ induces an injection of $\pi_1(M)$ into $\pi_1(\hat{M})$, $\rho_i \in AH(M)$. We next observe that one can choose \hat{N} so that N_{ρ_0} and N_{ρ_1} are not isometric. Let $a_i = A_i \cap \partial_T \hat{M}$ and let a_i^* denote the geodesic representative of $\psi(a_i)$ in $\partial_c \hat{N}$. Notice that for each $i = 0, 1$ there is a conformal embedding of $\partial_c \hat{N} - a_i^*$ into $\partial_c N_{\rho_i}$ such that each component of the complement of the image of $\partial_c \hat{N} - a_i^*$ is a neighborhood of a cusp. One may therefore choose the conformal structure on $\partial_c \hat{N}$ so that there is not a conformal homeomorphism from $\partial_c N_{\rho_0}$ to $\partial_c N_{\rho_1}$. Therefore, N_{ρ_0} and N_{ρ_1} are not isometric.

Let $\{N_n = N_{\beta_n}\}$ be the sequence of hyperbolic 3-manifolds provided by the Hyperbolic Dehn Filling Theorem applied to the sequence $\{(1, n)\}_{n \in \mathbb{Z}_+}$

and let $\{\psi_n : \hat{M}(1, n) - \partial_T \hat{M}(1, n) \rightarrow N_n \cup \partial_c N_n\}$ be the homeomorphisms such that $\psi_n \circ i_n \circ \psi^{-1}$ is conformal on $\partial_c N$. Let

$$\rho_{n,i} = \beta_n \circ \rho_i$$

for all n large enough that N_n and ψ_n exist. Since $\beta_n \circ \psi_*$ is conjugate to $(\psi_n \circ i_n)_*$ (by applying part (2) of the Hyperbolic Dehn Filling Theorem) and $i_n \circ h_i$ is homotopic to a homeomorphism, we see that $\rho_{n,i} = (\psi_n \circ i_n \circ h_i)_*$ lies in $AH(M)$ for all n and each i . It follows from part (1) of the Hyperbolic Dehn Filling Theorem that $\{\rho_{n,i}\}$ converges to ρ_i for each i . Moreover, $\rho_{n,1} = \rho_{n,0} \circ (D_A)_*$ for all n , since $i_n \circ h_1$ is homotopic to $i_n \circ h_0 \circ D_A^n$ for all n .

Therefore, $\{\rho_{n,0}\}$ and $\{\rho_{n,1}\}$ project to the same sequence in $AI(M)$ and both N_{ρ_0} and N_{ρ_1} are limits of this sequence. Since N_{ρ_0} and N_{ρ_1} are distinct manifolds in $AI(M)$, it follows that $AI(M)$ is not Hausdorff. \square

Remark: One can also establish Theorem 1.2 using deformation theory of Kleinian groups and convergence results of Thurston [48]. This version of the argument follows the same outline as the proof of Proposition 3.3 in [17].

We provide a brief sketch of this argument. The classical deformation theory of Kleinian groups (in combination with Thurston's Hyperbolization Theorem) guarantees that there exists a component B of $\text{int}(AH(M))$ such that if $\rho \in B$, then there exists a homeomorphism $\bar{h}_\rho : M - \partial_T M \rightarrow N_\rho \cup \partial_c N_\rho$ and the point ρ is determined by the induced conformal structure on $\partial M - \partial_T M$. Moreover, every possible conformal structure on $\partial M - \partial_T M$ arises in this manner.

Let a_0 and a_1 denote the components of ∂A and let t_{a_0} and t_{a_1} denote Dehn twists about a_0 and a_1 respectively. We choose orientations so that D_A induces $t_{a_0} \circ t_{a_1}$ on ∂M . We then let $\rho_{n,0} \in B$ have associated conformal structure $t_{a_1}^n(X)$ and let $\rho_{n,1}$ have associated conformal structure $t_{a_0}^{-n}(X)$ for some conformal structure X on ∂M . Thurston's convergence results [47, 48] can be used to show that there exists a subsequence $\{n_j\}$ of \mathbb{Z} such that $\{\rho_{n_j,0}\}$ and $\{\rho_{n_j,1}\}$ both converge. One can guarantee, roughly as above, that the limiting hyperbolic manifolds are not isometric. Moreover, $\rho_{n,1} = \rho_{n,0} \circ (D_A)_*$ for all n , so we are in the same situation as in the proof above.

5. THE CHARACTERISTIC SUBMANIFOLD AND MAPPING CLASS GROUPS

In order to further analyze the case where M has incompressible boundary we will make use of the characteristic submanifold (developed by Jaco-Shalen [26] and Johannson [28]) and the theory of mapping class groups of 3-manifolds developed by Johannson [28] and extended by McCullough and his co-authors [37, 25, 16].

We begin by recalling the definition of the characteristic submanifold, specialized to the hyperbolic setting. In the general setting, the components of the characteristic submanifold are interval bundles and Seifert fibred spaces.

In the hyperbolic setting, the only Seifert fibred spaces which occur are the solid torus and the thickened torus (see Morgan [42, Sec. 11] or Canary-McCullough [16, Chap. 5]).

Theorem 5.1. *Let M be a compact oriented hyperbolizable 3-manifold with incompressible boundary. There exists a codimension zero submanifold $\Sigma(M) \subseteq M$ with frontier $Fr(\Sigma(M)) = \overline{\partial\Sigma(M)} - \partial M$ satisfying the following properties:*

- (1) *Each component Σ_i of $\Sigma(M)$ is either*
 - (i) *an interval bundle over a compact surface with negative Euler characteristic which intersects ∂M in its associated ∂I -bundle,*
 - (ii) *a thickened torus such that $\partial M \cap \Sigma_i$ contains a torus, or*
 - (iii) *a solid torus.*
- (2) *The frontier $Fr(\Sigma(M))$ is a collection of essential annuli and incompressible tori.*
- (3) *Any essential annulus or incompressible torus in M is properly isotopic into $\Sigma(M)$.*
- (4) *If X is a component of $M - \Sigma(M)$, then either $\pi_1(X)$ is non-abelian or \overline{X} is a solid torus which lies between an interval bundle component of $\Sigma(M)$ and a thickened or solid torus component of $\Sigma(M)$.*

Moreover, such a $\Sigma(M)$ is unique up to isotopy, and is called the characteristic submanifold of M .

The existence and the uniqueness of the characteristic submanifold in general follows from The Characteristic Pair Theorem in [26] or Proposition 9.4 and Corollary 10.9 in [28]. Theorem 5.1(1) follows from [16, Theorem 5.3.4], part (2) follows from (1) and the definition of the characteristic submanifold, part (3) follows from [28, Theorem 12.5], and part (4) follows from [16, Theorem 2.9.3].

Johannson's Classification Theorem [28] asserts that every homotopy equivalence between compact, irreducible 3-manifolds with incompressible boundary may be homotoped so that it preserves the characteristic submanifold and is a homeomorphism on its complement. Therefore, the study of $Out(\pi_1(M))$ often reduces to the study of mapping class groups of interval bundles and Seifert-fibered spaces.

Johannson's Classification Theorem [28, Theorem 24.2]. *Let M and Q be irreducible 3-manifolds with incompressible boundary and let $h : M \rightarrow Q$ be a homotopy equivalence. Then h is homotopic to a map $g : M \rightarrow Q$ such that*

- (1) $g^{-1}(\Sigma(Q)) = \Sigma(M)$,
- (2) $g|_{\Sigma(M)} : \Sigma(M) \rightarrow \Sigma(Q)$ is a homotopy equivalence, and
- (3) $g|_{\overline{M - \Sigma(M)}} : \overline{M - \Sigma(M)} \rightarrow \overline{Q - \Sigma(Q)}$ is a homeomorphism.

Moreover, if h is a homeomorphism, then g is a homeomorphism.

We let the mapping class group $\text{Mod}(M)$ denote the group of isotopy classes of self-homeomorphisms of M . Since M is irreducible and has (non-empty) incompressible boundary, any two homotopic homeomorphisms are isotopic (see Waldhausen [51, Theorem 7.1]), so $\text{Mod}(M)$ is naturally a subgroup of $\text{Out}(\pi_1(M))$. For simplicity, we will assume that M is a compact hyperbolizable 3-manifold with incompressible boundary and no toroidal boundary components. Notice that this implies that $\Sigma(M)$ contains no thickened torus components. Let Σ be the characteristic submanifold of M and denote its components by $\{\Sigma_1, \dots, \Sigma_k\}$.

Following McCullough [37], we let $\text{Mod}(\Sigma_i, Fr(\Sigma_i))$ denote the group of homotopy classes of homeomorphisms $h : \Sigma_i \rightarrow \Sigma_i$ such that $h(F) = F$ for each component F of $Fr(\Sigma_i)$. We let $G(\Sigma_i, Fr(\Sigma_i))$ denote the subgroup consisting of (homotopy classes of) homeomorphisms which have representatives which are the identity on $Fr(\Sigma_i)$. Define $G(\Sigma, Fr(\Sigma)) = \bigoplus_{i=1}^k G(\Sigma_i, Fr(\Sigma_i))$. Notice that using these definitions, the restriction of a Dehn twist along a component of $Fr(\Sigma)$ is trivial in $G(\Sigma, Fr(\Sigma))$. There is a natural homomorphism of $G(\Sigma, Fr(\Sigma))$ into $\text{Mod}(M)$ (and hence into $\text{Out}(\pi_1(M))$). Corollary 18.2 in Johannson [28] (see also [16, Theorem 2.1.4]) implies that this homomorphism is injective, so we may think of $G(\Sigma, Fr(\Sigma))$ as a subgroup of $\text{Out}(\pi_1(M))$.

In our case, each Σ_i is either an interval bundle over a compact surface F_i with negative Euler characteristic or a solid torus. If Σ_i is an interval bundle over a compact surface F_i , then $G(\Sigma_i, Fr(\Sigma_i))$ is isomorphic to $\text{Mod}(F_i, \partial F_i)$ (see Proposition 3.2.1 in [37] and Lemma 6.1 in [25]). If Σ_i is a solid torus, then $G(\Sigma_i, Fr(\Sigma_i))$ is finite (see Lemma 10.3.2 in [16]).

Let $J(M)$ be the subgroup of $\text{Mod}(M)$ consisting of classes represented by homeomorphisms fixing $M - \Sigma$ pointwise. Lemma 4.2.1 of McCullough [37] implies that $J(M)$ has finite index in $\text{Mod}(M)$. (Instead of $J(M)$, McCullough writes $\mathcal{K}(M, \Sigma_1, \Sigma_2, \dots, \Sigma_k)$.) Lemma 4.2.2 of McCullough [37] implies that the kernel $K(M)$ of the natural surjective homomorphism

$$p_\Sigma : J(M) \rightarrow G(\Sigma, Fr(\Sigma))$$

is abelian and is generated by Dehn twists about the annuli in $Fr(\Sigma)$.

We summarize the discussion above in the following statement.

Theorem 5.2. *Let M be a compact hyperbolizable 3-manifold with incompressible boundary and no toroidal boundary components. Then there is a finite index subgroup $J(M)$ of $\text{Mod}(M)$ and a split exact sequence*

$$1 \longrightarrow K(M) \longrightarrow J(M) \xrightarrow{p_\Sigma} G(\Sigma, Fr(\Sigma)) \longrightarrow 1$$

such that $K(M)$ is an abelian group generated by Dehn twists about essential annuli in $Fr(\Sigma)$.

6. CHARACTERISTIC COLLECTIONS OF ANNULI

We continue to assume that M has incompressible boundary and no toroidal boundary components and that $\Sigma(M)$ is its characteristic submanifold. In this section, we organize $K(M)$ into subgroups generated by collections of annuli with homotopic core curves, called characteristic collection of annuli, and define a class of free subgroups of $\pi_1(M)$ which “register” these subgroups of $K(M)$.

A *characteristic collection of annuli* for M is either a) the collection of all frontier annuli in a solid torus component of $\Sigma(M)$, or b) an annulus in the frontier of an interval bundle component of $\Sigma(M)$ which is not properly isotopic to a frontier annulus of a solid torus component of $\Sigma(M)$. Johannson’s Classification Theorem implies that if $\{C_1, \dots, C_m\}$ is the set of characteristic collection of annuli and $\varphi \in \text{Out}(\pi_1(M))$, then there exists a homotopy equivalence $h : M \rightarrow M$ such that $h_* = \varphi$ and $h(C_j)$ is a characteristic collection of annuli for all j .

If C_j is a characteristic collection of annuli for M , let K_j be the subgroup of $K(M)$ generated by Dehn twists about the annuli in C_j . Notice that $K_i \cap K_j = \{id\}$ for $i \neq j$, since each element of K_j fixes any curve disjoint from C_j . Then $K(M) = \bigoplus_{j=1}^m K_j$, since every frontier annulus of $\Sigma(M)$ is properly isotopic to a component of some characteristic collection of annuli. Let $q_j : K(M) \rightarrow K_j$ be the projection map.

We next introduce free subgroups of $\pi_1(M)$, called C_j -registering subgroups, which are preserved by K_j and such that K_j acts effectively on the subgroup.

We first suppose that $C_j = \text{Fr}(T_j)$ where T_j is a solid torus component of $\Sigma(M)$. Let $\{A_1, \dots, A_l\}$ denote the components of $\text{Fr}(T_j)$. For each $i = 1, \dots, l$, let X_i be the component of $M - (T_j \cup C_1 \cup C_2 \cup \dots \cup C_m)$ abutting A_i . (Notice that each X_i is either a component of $M - \Sigma(M)$ or properly isotopic to the interior of an interval bundle component of $\Sigma(M)$.) For $x_0 \in T_j$, we say that a subgroup H of $\pi_1(M, x_0)$ is C_j -registering if it is freely (and minimally) generated by the core curve a of T_j and, for each $i = 1, \dots, l$, a loop g_i in $T_j \cup X_i$ based at x_0 intersecting A_i exactly twice. In particular, every C_j -registering subgroup of $\pi_1(M, x_0)$ is isomorphic to F_{l+1} .

Notice that a Dehn twist D_{A_i} along any A_i preserves H in $\pi_1(M, x_0)$. It acts on H by the map t_i which fixes a and g_m for $m \neq i$, and conjugates g_i by a^n (where the core curve of A_i is homotopic to a^n). Let $s_H : K_j \rightarrow \text{Out}(H)$ be the homomorphism which takes each D_{A_i} to t_i . Simultaneously twisting along all l annuli induces conjugation by a^n , which is an inner automorphism of H . Moreover, it is easily checked that $s_H(K_j)$ is isomorphic to \mathbf{Z}^{l-1} and is generated by t_1, \dots, t_{l-1} . The set $\{a, g_1, \dots, g_l\}$ may be extended to a generating set for $\pi_1(M, x_0)$ by appending curves which intersect $\text{Fr}(T_j)$ exactly twice, so $D_{A_1} \circ \dots \circ D_{A_l}$ acts as conjugation by a^n on all of $\pi_1(M, x_0)$. Therefore, K_j itself is isomorphic to \mathbf{Z}^{l-1} and s_H is injective. (In particular,

if C_j is a single annulus in the boundary of a solid torus component of $\Sigma(M)$, then K_j is trivial and we could have omitted C_j .)

Now suppose that $C_j = \{A\}$ is a frontier annulus of an interval bundle component Σ_i of Σ which is not properly isotopic into a solid torus component of Σ . Let x_0 be a point in A and let a be the core curve of A . We say that a subgroup H of $\pi_1(M, x_0)$ is C_j -registering if it is freely (and minimally) generated by a and two loops g_1 and g_2 based at x_0 each of whose interiors misses A , and which lie in the two distinct components of $M - (C_1 \cup C_2 \cup \dots \cup C_m)$ abutting A . In this case, H is isomorphic to F_3 . Arguing as above, it follows that K_j is an infinite cyclic subgroup of $Out(\pi_1(M))$ and there is an injection $s_H : K_j \rightarrow Out(H)$.

In either situation, if H is a C_j -registering group for a characteristic collection of annuli C_j , then we may consider the map

$$r_H : X(M) \rightarrow X(H)$$

simply obtained by taking ρ to $\rho|_H$. (Here, $X(H)$ is the $PSL_2(\mathbf{C})$ -character variety of the abstract group H .) One easily checks from the description above that if $\alpha \in K_j$, then $r_H(\rho \circ \alpha) = r_H(\rho) \circ s_H(\alpha)$ for all $\rho \in X(M)$. Notice that if $\varphi \in K_l$ and $j \neq l$, then K_l acts trivially on H , since each generating curve of H is disjoint from C_l . Therefore,

$$r_H(\rho \circ \alpha) = r_H(\rho) \circ s_H(q_j(\alpha))$$

for all $\rho \in X(M)$ and $\alpha \in K(M)$.

We summarize the key points of this discussion for use later:

Lemma 6.1. *Let M be a compact hyperbolizable 3-manifold with incompressible boundary and no toroidal boundary components. If C_j is a characteristic collection of annuli for M and H is a C_j -registering subgroup of $\pi_1(M)$, then H is preserved by each element of K_j and there is a natural injective homomorphism $s_H : K_j \rightarrow Out(H)$. Moreover, if $\alpha \in K(M)$, then $r_H(\rho \circ \alpha) = r_H(\rho) \circ s_H(q_j(\alpha))$ for all $\rho \in X(M)$.*

7. PRIMITIVE ESSENTIAL ANNULI AND MANIFOLDS WITH COMPRESSIBLE BOUNDARY

In this section we use a result of Johannson [28] to show that all compact hyperbolizable 3-manifolds with compressible boundary and no toroidal boundary components contain a primitive essential annulus. It then follows from Theorem 1.2 that if M has compressible boundary and no toroidal boundary components, then $AI(M)$ is not Hausdorff.

We first find indivisible curves in the boundary of compact hyperbolizable 3-manifolds with incompressible boundary and no toroidal boundary components. (We call a curve a in M *indivisible* if it generates a maximal abelian subgroup of $\pi_1(M)$.)

Lemma 7.1. *Let M be a compact hyperbolizable 3-manifold with (non-empty) incompressible boundary and no toroidal boundary components. Then there exists an indivisible simple closed curve in ∂M .*

Proof. We first observe that under our assumptions every maximal abelian subgroup of $\pi_1(M)$ is cyclic (since every non-cyclic abelian subgroup of the fundamental group of a compact hyperbolizable 3-manifold is conjugate into the fundamental group of a toroidal component of ∂M , see [42, Corollary 6.10]). Theorem 32.1 in Johannson [28] (see also Jaco-Shalen [27]) implies that an essential simple closed curve in ∂M which is not indivisible is isotopic into $\Sigma(M) \cap \partial M$ and if it is isotopic into an interval bundle component Σ_i of M , then it is isotopic to a boundary component of an essential Möbius band in Σ_i . If $\Sigma(M)$ is not all of M , then any simple closed curve in ∂M which cannot be isotoped into $\Sigma(M)$ is indivisible.

If $\Sigma(M) = M$, then M is an interval bundle over a closed surface with negative Euler characteristic and the proof is completed by the following lemma, whose full statement will be used later in the paper.

Lemma 7.2. *Let M be a compact hyperbolizable 3-manifold with no toroidal boundary components. Let Σ_i be an interval bundle component of $\Sigma(M)$, then there is a primitive essential annulus (for M) contained in Σ_i .*

Proof. Let F_i be the base surface of Σ_i and let a be a non-peripheral simple closed curve in F_i which is two-sided, homotopically non-trivial and does not bound a Möbius band. Then a is an indivisible curve in F_i and hence in M . The sub-interval bundle A over a is thus a primitive essential annulus. \square

\square

We are now prepared to prove the main result of the section.

Proposition 7.3. *If M is a compact hyperbolizable 3-manifold with compressible boundary and no toroidal boundary components, then M contains a primitive essential annulus.*

Proof. We first suppose that M is a compression body. If M is a handlebody, then it is an interval bundle, so contains a primitive essential annulus by Lemma 7.2. Otherwise, M is formed from $S \times I$ by appending 1-handles to $S \times \{1\}$, where S is a closed, but not necessarily connected, orientable surface. Let α be an essential simple closed curve in $S \times \{1\}$ which lies in ∂M . Let D be a disk in $S \times \{1\} - \partial M$. We may assume that α intersects ∂D in exactly one point. Let $\beta \subset (\partial M \cap S \times \{1\})$ be a simple closed curve homotopic to $\alpha * \partial D$ (in ∂M) and disjoint from α . Then α and β bound an embedded annulus in $S \times \{1\}$, which may be homotoped to a primitive essential annulus in M (by pushing the interior of the annulus into the interior of $S \times I$).

If M is not a compression body, let C_M be a characteristic compression body neighborhood of ∂M (as discussed in Section 2). Let C be a component

of C_M which has a compressible boundary component ∂_+C and an incompressible boundary component F . Let X be the component of $\overline{M - C_M}$ which contains F in its boundary and let α be an essential simple closed curve in F which is indivisible in X (which exists by Lemma 7.1). Let α' be a curve in $\partial_+C \subset \partial M$ which is homotopic to α . One may then construct as above a primitive essential annulus A in C with α' as one boundary component. It is clear that A remains essential in M . Since $\pi_1(M) = \pi_1(X) * H$ for some group H , the core curve of A , which is homotopic to α , is indivisible in $\pi_1(M)$. Therefore, A is our desired primitive essential annulus in M . \square

Remark: The above argument is easily extended to the case where M is allowed to have toroidal boundary components (but is still hyperbolizable), unless M is a compression body all of whose boundary components are tori. In fact, the only counterexamples in this situation occur when M is obtained from one or two untwisted interval bundles over tori by attaching exactly one 1-handle.

We have thus already established Corollary 1.4 in the case that M has compressible boundary.

Corollary 7.4. *If M is a compact hyperbolizable 3-manifold with compressible boundary, no toroidal boundary components, and non-abelian fundamental group, then the moduli space $AI(M)$ is not Hausdorff.*

8. THE SPACE $AH_n(M)$

In this section, we continue to assume that M has incompressible boundary and no toroidal boundary components. We study the space $AH_n(M)$ of representations $\rho \in AH(M)$ such that $\rho(\pi_1(\Sigma(M)))$ is purely hyperbolic. Formally, $AH_n(M)$ is the set of (conjugacy classes of) representations $\rho \in AH(M)$ such that if g is a non-trivial element of $\pi_1(M)$ which is conjugate into $\pi_1(\Sigma_i)$ for a component Σ_i of $\Sigma(M)$, then $\rho(g)$ is hyperbolic. We will see later that $Out(\pi_1(M))$ acts properly discontinuously on an open neighborhood of $AH_n(M)$ in $X(M)$ if M is not an interval bundle.

We observe that $\text{int}(AH(M))$ is a proper subset of $AH_n(M)$ and that $AH(M) = AH_n(M)$ if and only if M contains no primitive essential annuli.

Lemma 8.1. *Let M be a compact hyperbolizable 3-manifold with incompressible boundary and no toroidal boundary components. Then*

- (1) *the interior of $AH(M)$ is a proper subset of $AH_n(M)$, and*
- (2) *$AH_n(M) = AH(M)$ if and only if M contains no primitive essential annuli.*

Proof. Sullivan [46] proved that all representations in $\text{int}(AH(M))$ are purely hyperbolic (if M has no toroidal boundary components), so clearly $\text{int}(AH(M))$ is contained in $AH_n(M)$. On the other hand, there always exist purely hyperbolic $\rho \in \partial AH(M)$ (see, for example, Lemmas 4.1 and 4.2 in Canary-Hersonsky [15]), so $\text{int}(AH(M))$ is a proper subset of $AH_n(M)$.

If M contains a primitive essential annulus A , then there exist $\rho \in AH(M)$ such that $\rho(\alpha)$ is parabolic (where α is the core curve of A), so $AH_n(M)$ is not all of $AH(M)$ in this case (see Ohshika [43]).

Now suppose that M contains no primitive essential annuli. We first note that every component of $\Sigma(M)$ is a solid torus, since any interval bundle component of M contains a primitive essential annulus (by Lemma 7.2). Let T be a solid torus component of $\Sigma(M)$. A frontier annulus A of T is an essential annulus in M , so it must not be primitive. It follows that the core curve a of T is not peripheral in M (see [28, Theorem 32.1]).

Let $\rho \in AH(M)$ and let R be a relative compact core for $(N_\rho)_\epsilon^0$ (for some $\epsilon < \mu$). Let $h : M \rightarrow R$ be a homotopy equivalence in the homotopy class determined by ρ . By Johannson's Classification Theorem [28, Thm.24.2], h may be homotoped so that $h(T)$ is a component T' of $\Sigma(R)$, $h|_{Fr(T)}$ is an embedding with image $Fr(T')$ and $h|_T : (T, Fr(T)) \rightarrow (T', Fr(T'))$ is a homotopy equivalence of pairs. It follows that $h(a)$ is homotopic to the core curve of T' which is not peripheral in R .

If $\rho(a)$ were parabolic, then $h(a)$ would be homotopic into a non-compact component of $(N_\rho)_{thin(\epsilon)}$ and hence into $P = R \cap \partial(N_\rho)_\epsilon^0 \subset \partial R$, so $h(a)$ would be peripheral in R . It follows that $\rho(a)$ is hyperbolic. Since a generates $\pi_1(T)$, we see that $\rho(\pi_1(T))$ is purely hyperbolic. Since T is an arbitrary component of $\Sigma(M)$, we see that $\rho \in AH_n(M)$. \square

We next check that the restriction of $\rho \in AH_n(M)$ to the fundamental group of an interval bundle component of $\Sigma(M)$ is Schottky. By definition, a Schottky group is a free, geometrically finite, purely hyperbolic subgroup of $PSL_2(\mathbf{C})$ (see Maskit [34] for a discussion of the equivalence of this definition with more classical definitions).

Lemma 8.2. *Let M be a compact hyperbolizable 3-manifold with incompressible boundary with no toroidal boundary components which is not an interval bundle. If Σ_i is an interval bundle component of $\Sigma(M)$ and $\rho \in AH_n(M)$, then $\rho(\pi_1(\Sigma_i))$ is a Schottky group.*

Proof. By definition $\rho(\pi_1(\Sigma_i))$ is purely hyperbolic, so it suffices to prove it is free and geometrically finite. Let R be a compact core of the hyperbolic manifold N_ρ and let $h : M \rightarrow R$ be a homotopy equivalence in the homotopy class associated to ρ . Johannson's Classification Theorem [28, Thm.24.2] implies that we may choose h and an interval bundle component Σ_j of $\Sigma(R)$ such that $h(\Sigma_i) = \Sigma_j$, h restricts to a homeomorphism from $Fr(\Sigma_i)$ to $Fr(\Sigma_j)$ and $h|_{\Sigma_i} : \Sigma_i \rightarrow \Sigma_j$ is a homotopy equivalence. Since M is not an interval bundle, Σ_i is an interval bundle over a compact surface with boundary, so Σ_j is also an interval bundle over a compact surface with boundary, hence a handlebody. In particular, $\pi_1(\Sigma_i)$ and hence $\rho(\pi_1(\Sigma_i))$ is free.

Let $p_j : N_j \rightarrow N_\rho$ be the cover of N_ρ associated to $\rho(\pi_1(\Sigma_i)) = \pi_1(\Sigma_j)$. Since M is not an interval bundle, $\pi_1(\Sigma_j)$ is an infinite index subgroup of

$\pi_1(M)$, so p_j is an infinite-to-one covering. The interval bundle Σ_j lifts to a compact core $\tilde{\Sigma}_j$ for N_j . Since $\rho(\pi_1(\Sigma_i))$ is purely hyperbolic, $\tilde{\Sigma}_j$ is also a relative compact core for N_j . Therefore, $N_j = (N_j)_\epsilon^0$ has one end and p_j is infinite-to-one on this end, so the Covering Theorem (see [14]) implies that this end is geometrically finite, and hence that N_j is geometrically finite. Therefore, $\rho(\pi_1(\Sigma_i))$ is geometrically finite, completing the proof that it is a Schottky group. \square

Finally, we check that if $\rho \in AH_n(M)$ and C_j is a characteristic collection of annuli, then there exists a C_j -registering subgroup whose image under ρ is Schottky.

Lemma 8.3. *Suppose that M is a compact hyperbolizable 3-manifold with incompressible boundary and no toroidal boundary components and C_j is a characteristic collection of frontier annuli for M . If $\rho \in AH_n(M)$, then there exists a C_j -registering subgroup H of $\pi_1(M)$ such that $\rho(H)$ is a Schottky group.*

Proof. We first suppose that $C_j = \{A\}$ is a frontier annulus of an interval bundle component of $\Sigma(M)$ (and that A is not properly isotopic to a frontier annulus of a solid torus component of $\Sigma(M)$) and let $x_0 \in A$. We identify $\pi_1(M)$ with $\pi_1(M, x_0)$. Let X_1 and X_2 be the (distinct) components of $M - Fr(\Sigma)$ abutting A . Notice that each X_i must have non-abelian fundamental group, since it either contains (the interior of) an interval bundle component of $\Sigma(M)$ or (the interior of) a component of $M - \Sigma(M)$ which is not a solid torus lying between an interval bundle component of $\Sigma(M)$ and a solid torus component of $\Sigma(M)$.

Let a be the core curve of A (based at x_0). Let F be a fundamental domain for the action of $\langle \rho(a) \rangle$ on $\Omega(\langle \rho(a) \rangle)$ which is an annulus in $\hat{\mathbb{C}}$. Since each $\rho(\pi_1(\overline{X}_i, x_0))$ is discrete, torsion-free and non-abelian, hence non-elementary, we may choose hyperbolic elements $\gamma_i \in \rho(\pi_1(\overline{X}_i, x_0))$ whose fixed points lie in the interior of F . There exists $s > 0$ such that one may choose (round) disks $D_i^\pm \subset \text{int}(F)$ about the fixed points of γ_i , such that $\gamma_i^s(\text{int}(D_i^-)) = \hat{\mathbb{C}} - D_i^+$, and D_1^+, D_1^-, D_2^+ and D_2^- are disjoint. Then, the Klein Combination Theorem (commonly referred to as the ping pong lemma), guarantees that $\rho(a)$, γ_1^s and γ_2^s freely generate a Schottky group, see, for example, Theorem C.2 in Maskit [35]. Then each $\rho^{-1}(\gamma_i^s)$ is represented by a curve g_i in \overline{X}_i based at x_0 and a , g_1 and g_2 generate a C_j -registering subgroup H such that $\rho(H)$ is Schottky.

Now suppose that $C_j = \{A_1, \dots, A_l\}$ is the collection of frontier annuli of a solid torus component T_j of $\Sigma(M)$. Let X_i be the component of $M - (T_j \cup C_1 \cup \dots \cup C_m)$ abutting A_i . Pick x_0 in T_j and let a be a core curve of T_j passing through x_0 . Again each X_i must have non-abelian fundamental group.

Let F be an annular fundamental domain for the action of $\langle \rho(a) \rangle$ on the complement in $\hat{\mathbb{C}}$ of the fixed points of $\rho(a)$. For each i , let $Y_i = X_i \cup A_i \cup \text{int}(T_j')$

and pick a hyperbolic element γ_i in $\rho(\pi_1(Y_i, x_0))$ both of whose fixed points lie in the interior of F . (Notice that even though it could be the case that $X_i = X_k$ for $i \neq k$, we still have that $\pi_1(Y_i, x_0)$ intersects $\pi_1(Y_k, x_0)$ only in the subgroup generated by a , so these hyperbolic elements are all distinct.) Then, just as in the torus case, there exists $s > 0$ such that the elements $\{\rho(a), \gamma_1^s, \dots, \gamma_l^s\}$ freely generate a Schottky group. Each $\rho^{-1}(\gamma_i^s)$ can be represented by a loop g_i based at x_0 which lies in Y_i and intersects A_i exactly twice. Therefore, the group H generated by $\{a, g_1, \dots, g_l\}$ is C_j -registering and $\rho(H)$ is Schottky. \square

9. PROPER DISCONTINUITY ON $AH_n(M)$

We are finally prepared to prove that $Out(\pi_1(M))$ acts properly discontinuously on an open neighborhood of $AH_n(M)$ if M is a compact hyperbolizable 3-manifold with incompressible boundary and no toroidal boundary components which is not an interval bundle.

Theorem 9.1. *Let M be a compact hyperbolizable 3-manifold with nonempty incompressible boundary and no toroidal boundary components which is not an interval bundle. Then there exists an open $Out(\pi_1(M))$ -invariant neighborhood $W(M)$ of $AH_n(M)$ in $X(M)$ such that $Out(\pi_1(M))$ acts properly discontinuously on $W(M)$.*

Notice that Theorem 1.3 is an immediate consequence of Proposition 7.3, Lemma 8.1 and Theorem 9.1. Moreover, Theorem 1.5 is an immediate corollary of Lemma 8.1 and Theorem 9.1.

We now provide a brief outline of the section. In 9.1 we recall Minsky's work which shows that $Out(\pi_1(H_n))$ acts properly discontinuously on the open set $PS(H_n)$ of primitive-stable representations in $X(H_n)$ where H_n is the handlebody of genus g . In 9.2, we use Minsky's work to show that if $V(M)$ is the set of all representations $\rho \in X(M)$ such that $\rho|_{\pi_1(\Sigma_i)}$ is primitive-stable for every interval bundle component Σ_i of $\Sigma(M)$, then $V(M)$ is $Out(\pi_1(M))$ -invariant and $G(\Sigma, Fr(\Sigma))$ acts properly discontinuously on $V(M)$. In 9.3, we consider the set $Z(M) \subset X(M)$ such that if $\rho \in Z(M)$ and C_j is a characteristic collection of annuli, then there exists a C_j -registering subgroup H of $\pi_1(M)$ such that $\rho|_H$ is primitive stable. We show that $Z(M)$ is $Out(\pi_1(M))$ -invariant and $K(M)$ acts properly discontinuously on $V(M)$. In 9.4, we let $W(M) = Z(M) \cap V(M)$ and use (slightly stronger versions of) the above results to show that $J(M)$ acts properly discontinuously on $W(M)$. Since $J(M)$ has finite index in $Out(\pi_1(M))$, this immediately implies Theorem 9.1.

9.1. Schottky groups and primitive-stable groups. In this section, we recall Minsky's work [41] on primitive-stable representations of the free group F_n , where $n \geq 2$. An element of F_n is called *primitive* if it is an element of a minimal free generating set for F_n . Let X be a bouquet of n

circles with base point b and fix a specific identification of $\pi_1(X, b)$ with F_n . To a conjugacy class $[w]$ in F_n one can associated an infinite geodesic in X which is obtained by concatenating infinitely many copies of a cyclically reduced representative of w (here the cyclic reduction is in the generating set associated to the natural generators of $\pi_1(X, b)$). Let \mathcal{P} denote the set of infinite geodesics in the universal cover \tilde{X} of X which project to geodesics associated to primitive words of F_n .

Given a representation $\rho : F_n \rightarrow \mathrm{PSL}_2(\mathbf{C})$, $x \in \mathbb{H}^3$ and a lift \tilde{b} of b , one obtains a unique ρ -equivariant map $\tau_{\rho, x} : \tilde{X} \rightarrow \mathbb{H}^3$ which takes \tilde{b} to x and maps each edge of \tilde{X} to a geodesic. A representation $\rho : F_n \rightarrow \mathrm{PSL}_2(\mathbf{C})$ is *primitive-stable* if there are constants $K, \delta > 0$ such that $\tau_{\rho, x}$ takes all the geodesics in \mathcal{P} to (K, δ) -quasi-geodesics in \mathbb{H}^3 . We let $PS(H_n)$ denote the set of (conjugacy classes) of primitive-stable representations in $X(H_n)$ where H_n is the handlebody of genus n .

We summarize the key points of Minsky's work which we use in the remainder of the section. We recall that Schottky space $\mathcal{S}_n \subset X(H_n)$ is the space of discrete faithful representations whose image is a Schottky group and that \mathcal{S}_n is the interior of $AH(H_n)$.

Theorem 9.2. (Minsky [41]) *If $n \geq 2$, then*

- (1) *$Out(F_n)$ acts properly discontinuously on $PS(H_n)$,*
- (2) *$PS(H_n)$ is an open subset of $X(H_n)$, and*
- (3) *Schottky space \mathcal{S}_n is a proper subset of $PS(H_n)$.*

Moreover, if $\tau \in PS(H_n)$, then there exists an open neighborhood U of τ in $PS(H_n)$ such that if $\{\alpha_n\}$ is a sequence of distinct elements of $Out(F_n)$, then $\{\alpha_n(U)\}$ exits every compact subset of $X(H_n)$ (i.e. for any compact subset C of $X(H_n)$ there exists N such that if $n \geq N$, then $\alpha_n(U) \cap C = \emptyset$).

Remark: In order to prove our main theorem it would suffice to use Schottky space \mathcal{S}_n in place of $PS(H_n)$. However, the subset $W(M)$ we obtain using $PS(H_n)$ is larger than we would obtain using simply \mathcal{S}_n .

9.2. Interval bundle components of $\Sigma(M)$. We will assume for the remainder of the section that M is a compact hyperbolizable 3-manifold with incompressible boundary and no toroidal boundary components which is not an interval bundle. Main Topological Theorem 2 in Canary and McCullough [16] (which is itself an exercise in applying Johannson's theory) implies that that if M has incompressible boundary and no toroidal boundary components, then $\mathrm{Mod}(M)$ has finite index in $Out(\pi_1(M))$. Therefore, applying Theorem 5.2, we see that $J(M)$ has finite index in $Out(\pi_1(M))$.

Let Σ_i be an interval bundle component of $\Sigma(M)$ with base surface F_i and let $X(\Sigma_i)$ be its associated character variety. There exists a natural restriction map $r_i : X(M) \rightarrow X(\Sigma_i)$ taking ρ to $\rho|_{\pi_1(\Sigma_i)}$. Recall that $G(\Sigma_i, Fr(\Sigma_i))$ is naturally identified with $\mathrm{Mod}(F_i, \partial F_i) \subset Out(\pi_1(\Sigma_i))$, and so acts effectively on $X(\Sigma_i)$. Moreover, if $\alpha \in J(M)$, then $r_i(\rho \circ \alpha) = r_i(\rho) \circ p_i(\alpha)$ where

p_i is the projection of $J(M)$ onto $G(\Sigma_i, Fr(\Sigma_i))$. We define

$$V(\Sigma_i) = r_i^{-1}(PS(\Sigma_i)).$$

Lemma 8.2 implies that $AH_n(M) \subset V(\Sigma_i)$ and $V(\Sigma_i)$ is open since r_i is continuous. Theorem 9.2 implies that $G(\Sigma_i, Fr(\Sigma_i))$ acts properly discontinuously on $V(\Sigma_i)$. In fact, we obtain the following somewhat stronger statement.

Lemma 9.3. *Let M be a compact hyperbolizable 3-manifold with nonempty incompressible boundary and no toroidal boundary components which is not an interval bundle. If Σ_i is an interval bundle component of Σ and $\rho \in V(\Sigma_i)$, then there exists an open neighborhood U of ρ in $V(\Sigma_i)$ such that if $\{\alpha_n\}$ is a sequence in $J(M)$ such that $\{p_i(\alpha_n)\}$ is sequence of distinct elements of $G(\Sigma_i, Fr(\Sigma_i))$, then $\{\alpha_n(U)\}$ exits every compact subset of $X(M)$.*

If $\{\Sigma_1, \dots, \Sigma_n\}$ denotes the collection of all interval bundle components of $\Sigma(M)$, then we let

$$V(M) = \bigcap_{i=1}^n V(\Sigma_i).$$

Lemma 9.3 implies that $G(\Sigma, Fr(\Sigma))$ acts properly discontinuously on $V(M)$. Moreover, we obtain:

Lemma 9.4. *Let M be a compact hyperbolizable 3-manifold with nonempty incompressible boundary and no toroidal boundary components which is not an interval bundle. Then*

- (1) $V(M)$ is an $Out(\pi_1(M))$ -invariant open neighborhood of $AH_n(M)$ in $X(M)$, and
- (2) if $\rho \in V(M)$, then there exists an open neighborhood U of ρ in $V(M)$ such that if $\{\alpha_n\}$ is a sequence in $J(M)$ such that $\{p_\Sigma(\alpha_n)\}$ is a sequence of distinct elements of $G(\Sigma, Fr(\Sigma))$, then $\{\alpha_n(U)\}$ exits every compact subset of $X(M)$.

Proof. Johannson's Classification Theorem implies that if $\varphi \in Out(\pi_1(M))$, and Σ_i is an interval bundle component of $\Sigma(M)$, then $\varphi(\pi_1(\Sigma_i))$ is conjugate to $\pi_1(\Sigma_j)$ where Σ_j is also an interval bundle component of $\Sigma(M)$. It follows that $V(M)$ is invariant under $Out(\pi_1(M))$. Since each $V(\Sigma_i)$ is open and contains $AH_n(M)$, $V(M)$ is an open neighborhood of $AH_n(M)$ in $X(M)$, completing the proof of claim (1).

Suppose that $\rho \in V$. Then, for each $i = 1, \dots, n$, let U_i be an open neighborhood of ρ in $V(\Sigma_i)$ such that if $\{\beta_n\}$ is a sequence in $J(M)$ such that $\{p_i(\beta_n)\}$ is a sequence of distinct elements of $G(\Sigma_i, Fr(\Sigma_i))$, then $\{\beta_n(U_i)\}$ exits every compact subset of $X(M)$ (see Lemma 9.3). Let $U = \bigcap_{i=1}^n U_i$.

Let $\{\alpha_n\}$ be a sequence of elements of $J(M)$ such that $\{p_\Sigma(\alpha_n)\}$ is a sequence of distinct elements of $G(\Sigma, Fr(\Sigma))$. Suppose that there exists a subsequence of $\{\alpha_n\}$ (still called $\{\alpha_n\}$) and a compact set $C \subset X(M)$ such that $\alpha_n(U) \cap C$ is nonempty for all n . Then there exists a subsequence,

still called $\{\alpha_n\}$, and an interval bundle component Σ_i of $\Sigma(M)$ such that $\{p_i(\alpha_n)\}$ is a sequence of distinct elements of $G(\Sigma_i, Fr(\Sigma_i))$ and $\alpha_n(U) \subset \alpha_n(U_i)$ intersects C for all n . (Recall that if Σ_k is a solid torus component of $\Sigma(M)$, then $G(\Sigma_k, Fr(\Sigma_k))$ is finite.) This contradicts the defining property of U_i and so establishes claim (2). \square

9.3. Characteristic collection of annuli. Let C_j be a characteristic collection of annuli in M . If H is a C_j -registering subgroup of $\pi_1(M)$, then the inclusion of H in $\pi_1(M)$ induces a natural injection $s_H : K_j \rightarrow Out(H)$ such that if $\alpha \in K(M)$, then $r_H(\rho \circ \alpha) = r_H(\rho) \circ s_H(q_j(\alpha))$ where $r_H(\rho) = \rho|_H$ (see Lemma 6.1). Let $Z_H = r_H^{-1}(PS(H))$ where $PS(H) \subset X(H)$ is the set of (conjugacy classes of) primitive-stable representations of H .

Let $Z(C_j) = \bigcup Z_H$ where the union is taken over all C_j -registering subgroups H of $\pi_1(M)$. Lemma 8.3 implies that $AH_n(M) \subset Z(C_j)$ and $Z(C_j)$ is open since each r_H is continuous. We again apply Theorem 9.2 to obtain the following lemma.

Lemma 9.5. *Let M be a compact hyperbolizable 3-manifold with nonempty incompressible boundary and no toroidal boundary components. Let C_j be a characteristic collection of annuli for M . Then $Z(C_j)$ is an open neighborhood of $AH_n(M)$ in $X(M)$ such that if $\rho \in Z(C_j)$, then there is an open neighborhood U of ρ in $Z(C_j)$ such that if $\{\alpha_n\}$ is a sequence in $K(M)$ such that $\{q_j(\alpha_n)\}$ is a sequence of distinct elements of K_j , then $\{\alpha_n(U)\}$ exits every compact subset of $X(M)$.*

If $\{C_1, \dots, C_m\}$ is the set of all characteristic collections of annuli for M , then we define

$$Z(M) = \bigcap_{i=1}^m Z(C_i).$$

As in the previous subsection, we may use Johannson's Classification Theorem and Lemma 9.5 to prove:

Lemma 9.6. *Let M be a compact hyperbolizable 3-manifold with nonempty incompressible boundary and no toroidal boundary components. Then*

- (1) $Z(M)$ is an $Out(\pi_1(M))$ -invariant open neighborhood of $AH_n(M)$ in $X(M)$, and
- (2) if $\rho \in Z(M)$, there is an open neighborhood U of ρ in $Z(M)$ such that if $\{\alpha_n\}$ is a sequence of distinct elements of $K(M)$, then $\alpha_n(U)$ exits every compact set of $X(M)$.

Proof. Johannson's Classification Theorem implies that if C_j is a characteristic collection of annuli for M and $\varphi \in Out(\pi_1(M))$, then there exists a homotopy equivalence $h : M \rightarrow M$ such that $h_* = \varphi$ and $h(C_j)$ is also a characteristic collection of annuli for M . Moreover, if H is a C_j -registering subgroup of $\pi_1(M)$, then $\varphi(H)$ is a $h(C_j)$ -registering subgroup of $\pi_1(M)$. Therefore, $Z(M)$ is $Out(\pi_1(M))$ -invariant. Since each $Z(C_j)$ is open and

contains $AH_n(M)$, $Z(M)$ is an open neighborhood of $AH_n(M)$ in $X(M)$, completing the proof of claim (1).

The proof of claim (2), follows the same outline as in the proof of part (2) in Lemma 9.4. \square

9.4. Assembly. Let $W(M) = V(M) \cap Z(M)$. Since $V(M)$ and $Z(M)$ are open $Out(\pi_1(M))$ -invariant neighborhoods of $AH_n(M)$, so is $W(M)$. It remains to prove that $Out(\pi_1(M))$ acts properly discontinuously on $W(M)$. Since $J(M)$ is a finite index subgroup of $Out(\pi_1(M))$, it suffices to prove that $J(M)$ acts properly discontinuously on $W(M)$.

Let $\rho \in W(M)$ and let U_ρ^V be the neighborhood of ρ guaranteed by Lemma 9.4 and let U_ρ^Z be the neighborhood of ρ provided by Lemma 9.6. Let $U_\rho = U_\rho^V \cap U_\rho^Z$ be the resulting open neighborhood of ρ (in $X(M)$) which is contained in $W(M)$.

We will show that if $\{\alpha_n\}$ is a sequence of distinct elements of $J(M)$ and $\rho \in W(M)$, then $\{\alpha_n(U_\rho)\}$ leaves every compact subset of $X(M)$, which suffices to establish proper discontinuity. We argue by contradiction.

Suppose that $\{\alpha_n\}$ is a sequence of distinct elements of $J(M)$ and $\rho \in W(M)$, such that $\{\alpha_n(U_\rho)\}$ does not leave every compact set. We may pass to a subsequence, still called $\{\alpha_n\}$, such that each $\alpha_n(U_\rho)$ intersects some fixed compact set C . If there is a further subsequence such that $\{p_\Sigma(\alpha_n)\}$ is a sequence of distinct elements of $G(\Sigma, Fr(\Sigma))$, then Lemma 9.4 guarantees that $\{\alpha_n(U_\rho^V)\}$ exits every compact subset of $X(M)$, which is a contradiction (since $\alpha_n(U_\rho) \subset \alpha_n(U_\rho^V)$).

Therefore, we may pass to a subsequence so that $p_\Sigma(\alpha_n) = \beta$ for all n and some fixed β . Thus, we can write $\alpha_n = \beta k_n$ where $\{k_n\}$ is a sequence of distinct elements in $K(M)$, since elements of $G(\Sigma, Fr(\Sigma))$ commute with elements of $K(M)$ (see Theorem 5.2). But then $\alpha_n(U_\rho) = \beta(k_n(U_\rho))$. Lemma 9.6 implies that $\{k_n(U_\rho^Z)\}$ exits every compact subset of $X(M)$, so $\{k_n(U_\rho)\}$ exits every compact subset of $X(M)$. The mapping class β induces a homeomorphism of $X(M)$, so $\{\alpha_n(U_\rho) = \beta(k_n(U_\rho))\}$ must also exit every compact subset of $X(M)$. This contradiction completes the proof of Theorem 9.1.

REFERENCES

- [1] I. Agol, “Tameness of hyperbolic 3-manifolds,” preprint.
- [2] J. W. Anderson and R. D. Canary, “Algebraic limits of Kleinian groups which rearrange the pages of a book,” *Invent. Math.* **126** (1996), 205–214.
- [3] J. W. Anderson, R. D. Canary and D. McCullough, “On the topology of deformation spaces of Kleinian groups,” *Annals of Math.* **152** (2000), 693–741.
- [4] R. Benedetti and C. Petronio, *Lectures on Hyperbolic Geometry*, Springer-Verlag Universitext, 1992.
- [5] L. Bers, “On moduli of Kleinian groups,” *Russian Math. Surveys* **29** (1974), 88–102.
- [6] F. Bonahon, “Cobordism of automorphisms of surfaces,” *Ann. Sci. École Norm. Sup.* **16** (1983), 237–270.
- [7] F. Bonahon and J. P. Otal, “Variétés hyperboliques à géodésiques arbitrairement courtes,” *Bull. L.M.S.* **20** (1988), 255–261.

- [8] B. Bowditch, “Markoff triples and quasi-Fuchsian groups,” *Proc. London Math. Soc.* **77** (1998), 697–736.
- [9] J. Brock and K. Bromberg, “On the density of geometrically finite Kleinian groups,” *Acta Math.* **192**(2004), 33–93.
- [10] K. Bromberg, “Rigidity of geometrically finite hyperbolic cone-manifolds,” *Geom. Ded.* **105**(2004), 143–170.
- [11] K. Bromberg “The space of Kleinian punctured torus groups is not locally connected,” preprint.
- [12] K. Bromberg and J. Holt, “Self-bumping of deformation spaces of hyperbolic 3-manifolds,” *J. Diff. Geom.* **57** (2001), 47–65.
- [13] D. Calegari and D. Gabai, “Shrinkwrapping and the taming of hyperbolic 3-manifolds,” *J. Amer. Math. Soc.* **19**(2006), 385–446.
- [14] R.D. Canary, “A covering theorem for hyperbolic 3-manifolds and its applications,” *Topology* **35**(1996), 751–778.
- [15] R.D. Canary and S. Hersonsky, “Ubiquity of geometric finiteness in boundaries of deformation spaces of hyperbolic 3-manifolds,” *Amer. J. Math.* **126**(2004), 1193–1220.
- [16] R.D. Canary and D. McCullough, *Homotopy equivalences of 3-manifolds and deformation theory of Kleinian groups*, Mem. Amer. Math. Soc. **172**(2004), no. 812.
- [17] R.D. Canary and Peter A. Storm, “The curious moduli spaces of unmarked Kleinian surface groups,” preprint.
- [18] S. Cantat, “Bers and Hénon, Painlevé and Schrödinger,” *Duke Math. J.* **149**(2009), 411–460.
- [19] T. D. Comar, *Hyperbolic Dehn surgery and convergence of Kleinian groups*, Ph.D. thesis, University of Michigan, 1996.
- [20] W. Goldman, “The complex-symplectic geometry of $SL_2(\mathbf{C})$ -characters over surfaces,” in *Algebraic groups and arithmetic*, Tata Inst. Fund. Res., Mumbai, 2004, 375–407.
- [21] W. Goldman, “Mapping class group dynamics on surface group representations,” *Problems on mapping class groups and related topics*, 189–214, Proc. Sympos. Pure Math., 74, *Amer. Math. Soc., Providence, RI*, 2006.
- [22] J. Hempel, *3-manifolds*, Princeton University Press, 1976.
- [23] M. Heusener and J. Porti, “The variety of characters in $PSL_2(\mathbf{C})$,” *Bol. Soc. Mat. Mex.*, **10**(2004), 221–238.
- [24] C. Hodgson and S.P. Kerckhoff, “Universal bounds for hyperbolic Dehn surgery,” *Annals of Math.* **162**(2005), 367–421.
- [25] S. Hong and D. McCullough, “Ubiquity of geometric finiteness in mapping class groups of Haken 3-manifolds,” *Pac. J. Math.* **188**(1999), 275–301.
- [26] W. Jaco and P. Shalen, *Seifert fibered spaces in 3-manifolds*, Mem. Amer. Math. Soc. **21** (1979), no. 220.
- [27] W. Jaco and P. Shalen, “Peripheral structure of 3-manifolds,” *Invent. Math.* **38**(1976), 55–87.
- [28] K. Johannson, *Homotopy Equivalences of 3-manifolds with Boundary*, Lecture Notes in Mathematics, vol. 761, Springer-Verlag, 1979.
- [29] M. Kapovich, *Hyperbolic manifolds and discrete groups*, Progr. Math. **183**, Birkhäuser, 2001.
- [30] S.P. Kerckhoff and W.P. Thurston, “Noncontinuity of the action of the modular group at Bers’ boundary of Teichmüller space,” *Invent. Math.* **100**(1990), 25–47.
- [31] R. Kulkarni and P. Shalen, “On Ahlfors’ finiteness theorem,” *Adv. in Math.* **76** (1989), 155–169.
- [32] M. Lee, in preparation.
- [33] A. Magid, *Deformation spaces of Kleinian surface groups are not locally connected*, Ph.D. thesis, University of Michigan, 2009.

- [34] B. Maskit, “On free Kleinian groups,” *Duke Math. J.* **48**(1981), 755–765.
- [35] B. Maskit, *Kleinian groups*, Springer-Verlag, 1988.
- [36] D. McCullough, “Compact submanifolds of 3-manifolds with boundary,” *Quart. J. Math. Oxford* **37** (1986), 299–307.
- [37] D. McCullough, “Virtually geometrically finite mapping class groups of 3-manifolds,” *J. Diff. Geom.* **33**(1991), 1–65.
- [38] D. McCullough and A. Miller, *Homeomorphisms of 3-manifolds with compressible boundary*, Mem. Amer. Math. Soc. **61**(1986), no. 344.
- [39] C. McMullen, *Renormalization and 3-manifolds which fiber over the circle*, Princeton University Press, 1996.
- [40] C. T. McMullen, “Complex earthquakes and Teichmüller theory,” *J. Amer. Math. Soc.* **11** (1998), 283–320.
- [41] Y. Minsky, “Note on dynamics of $Out(F_n)$ on $PSL_2(\mathbb{C})$ -characters,” preprint.
- [42] J. Morgan, “On Thurston’s uniformization theorem for three-dimensional manifolds” in *The Smith Conjecture*, Pure Appl. Math. **112**, Academic Press, 1984, 37–125.
- [43] K. Ohshika, “Geometrically finite Kleinian groups and parabolic elements,” *Proc. Edin. Math. Soc.* **41**(1998), 141–159.
- [44] P. Scott, “Compact submanifolds of 3-manifolds,” *J. London Math. Soc.* **7** (1974), 246–250.
- [45] J. Souto and P. Storm, “Dynamics of the mapping class group action on the variety of $PSL_2(\mathbb{C})$ -characters,” *Geom. Topol.* **10** (2006), 715–736.
- [46] D. Sullivan, “Quasiconformal homeomorphisms and dynamics. II. Structural stability implies hyperbolicity for Kleinian groups,” *Acta Math.* **155** (1985), 243–260.
- [47] W. P. Thurston, “Hyperbolic structures on 3-manifolds, II: Surface groups and 3-manifolds which fiber over the circle,” preprint.
- [48] W.P. Thurston, “Hyperbolic structures on 3-manifolds, III: Deformations of 3-manifolds with incompressible boundary,” preprint.
- [49] W. P. Thurston, *The Geometry and Topology of Three-Manifolds*, Princeton University course notes, available at <http://www.msri.org/publications/books/gt3m/> (1980).
- [50] S.P. Tan, Y. L. Wong, and Y. Zhang, “Generalized Markoff maps and McShane’s identity,” *Adv. Math.* **217**(2008), 761–813.
- [51] F. Waldhausen, “On irreducible 3-manifolds which are sufficiently large,” *Annals of Math.* **87** (1968), 56–88.