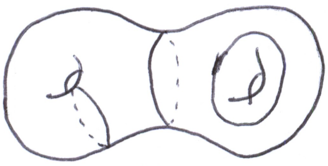
Laminations, Train tracks, & singular foliations June 10, 2009

Fix M a closed hyperbolic surface. (To keep things simple today we'll assume M has no boundary.)

Def: A geodesic lamination λ on M is a union of disjoint geodesics of M forming a closed set. (Note that a geodesic is either closed or bi-infinite.)

E.g.



a multicurve

a multicurve with additional "spiraling" geodesics

Each component of $M - \lambda$ has area $\geq \pi$, because it has geodesic boundary & one can apply Gauss-Bonnet.

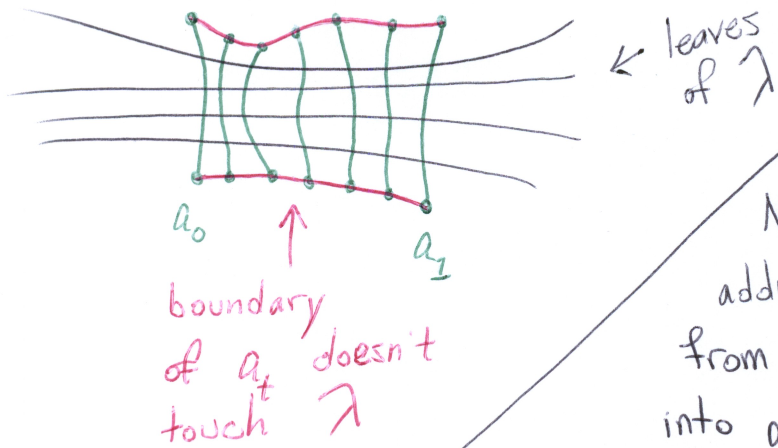
$\Rightarrow$ $M - \lambda$ has $\leq \dfrac{\text{Area}(M)}{\pi} = 2 \cdot |\chi(M)|$ components.

With a little more work one can show λ has area 0 in M. (For this and much more, see Chapter 8 section 5 of Thurston's Notes.)

With a homeomorphism $f: M \to N$, for N a hyperbolic surface, we can push λ to N by identifying a geodesic in M with a distinct pair in $\partial \tilde{M}$, & then using the π_1-equivariant homeom $\partial \tilde{M} \to \partial \tilde{N}$. So the choice of metric on M is just for convenience.

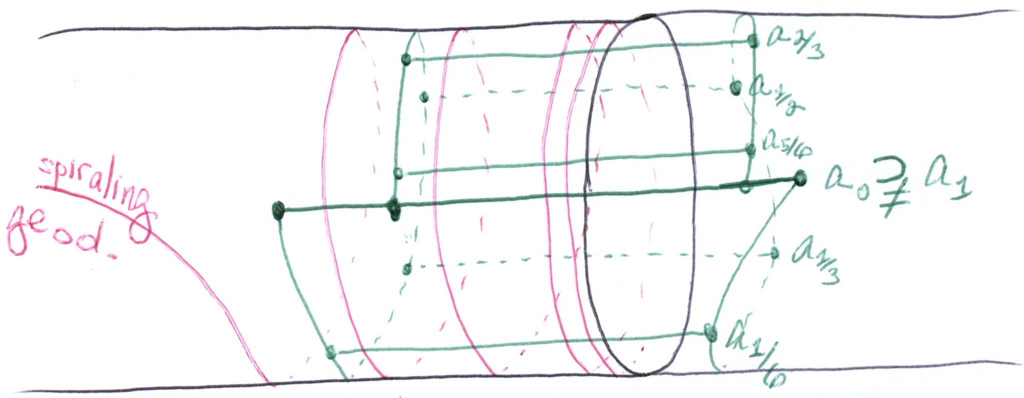
Def: A measured geodesic lamination on M is a geodesic lamination λ together with a measure μ on the set of compact arcs of M transverse to λ satisfying:

1. $\mu(a) < \infty$ for any compact arc a transverse to λ
2. If a_t is a 1-parameter family of compact arcs transverse to λ s.t. $\partial a_t \cap \lambda = \emptyset$ for all t then $\mu(a_0) = \mu(a_t)$ $\forall$ t.



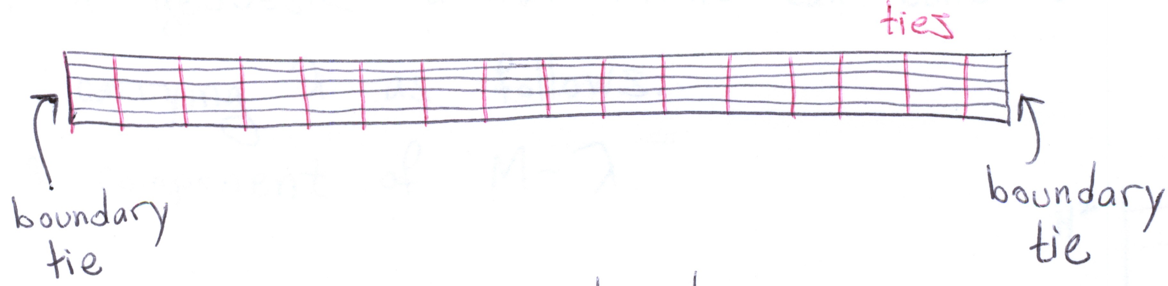
3. We assume μ has full support, i.e. $a \cap \lambda \neq \emptyset$ $\Rightarrow$ $\mu(a) > 0$.

Note the multicurve with additional "spiraling" geodesics from page 1 cannot be made into a measured geodesic lamination. A measured geod. lam'n cannot have an infinite geod. spiraling into a closed geod.

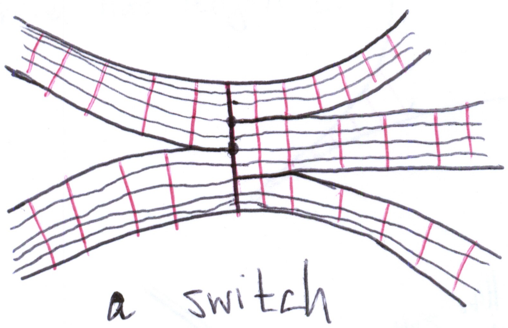


One could, in this situation, find a family a_t of transverse arcs s.t. $a_1 \subsetneq a_0$ $\Rightarrow$ $\mu(a_t) = 0$.

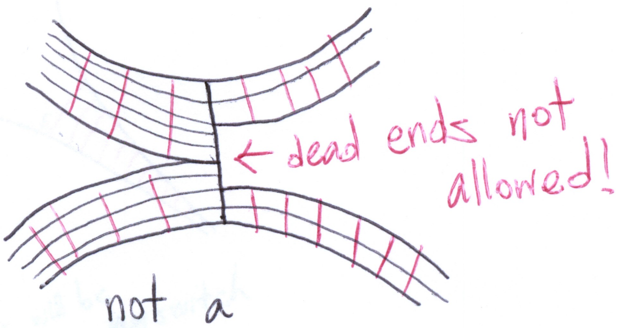
The next goal is to define train tracks. It's best to draw lots of pictures. A ~~piece~~ branch of track is an embedded square in M with ~~a~~ its vertical + horizontal foliation. The horizontal foliation forms the leaves of the ~~piece~~ branch. The vertical foliation form the ties.



A switch is a union of ~~pieces~~ branches glued along boundary ties so there are no dead-end leaves.

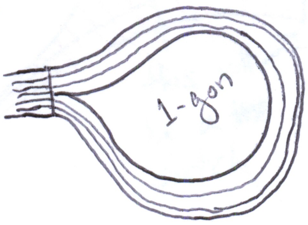


a switch

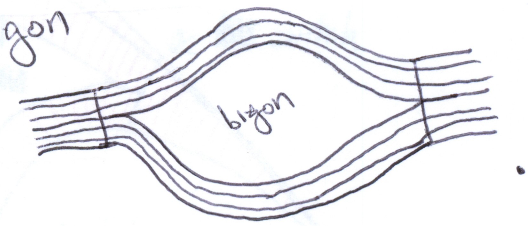


not a switch

A train track τ on M is a collection of ~~pieces~~ branches and switches so there are no dead-end leaves, and no component of $M-\tau$ is diffeom. to a ~~bigon~~ 1-gon e.g.

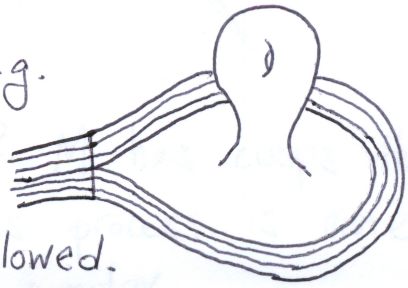


or a bigon



(or a 0-gon)

Note that topology in $M-\tau$ is allowed. E.g.



is allowed.

~~Note~~ Now assume λ is a geodesic lamination with measure μ. Note that inside any fixed ~~plane~~ branch of τ the measure of a tie is constant. Moreover, ~~thus~~ at a switch the total measure of the ties on the left equals the total measure of the ties on the right. This motivates the def'n

Def: A weighted train track assigns a positive weight to each branch such that at each switch the total weights on each side are equal.

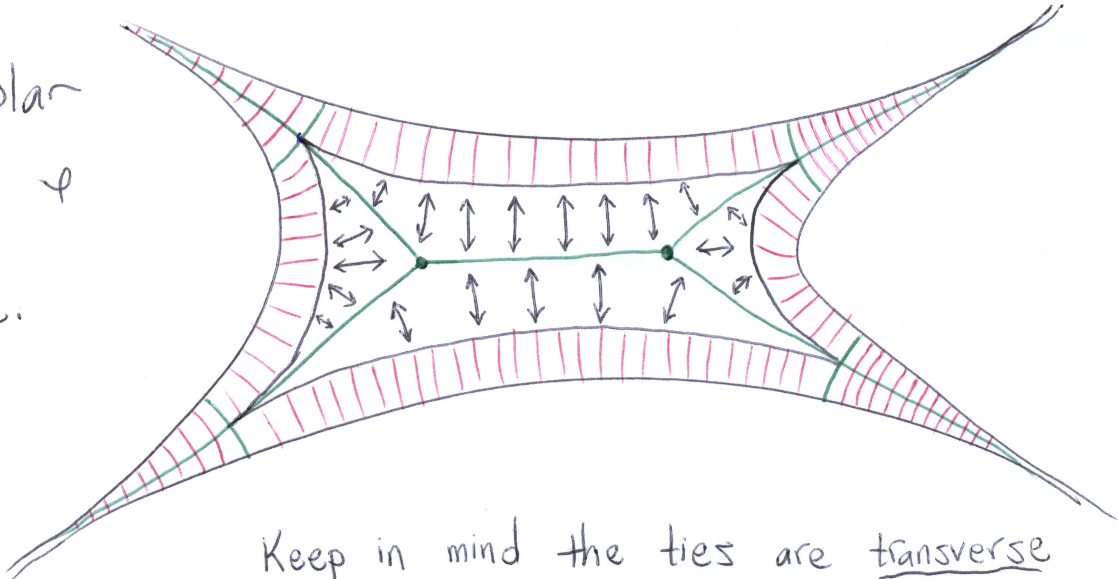
So our construction builds a weighted train track from a measured geodesic lamination.

Notice that by choosing ε smaller we obtain finer approximations of our lamination.

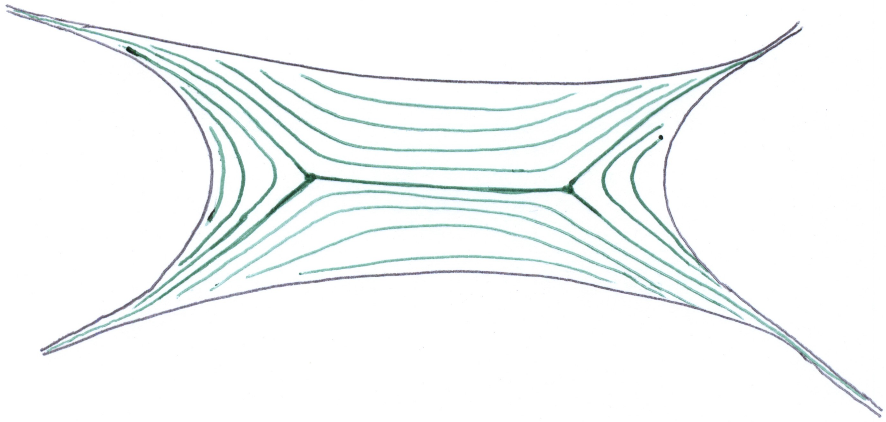
Building a measured singular foliation from a weighted train track is easy, if a bit technical to nail down. ~~Simply fill in the complementary regions collapse~~

First add some singular leaves to the complement of τ and then collapse the rest of the complement of τ onto the singular leaves.

Add singular leaves & then collapse.



Keep in mind the ties are <u>transverse</u> to the resulting foliation.



There is ambiguity when choosing how to add singular leaves. All choices are Whitehead equivalent. When a complementary region has some topology then adding singular leaves is slightly more complex. We'll skip these details here. This gives a singular foliation. What about the measure? For each branch of the train track of weight w put a uniform Lebesgue measure on the ties of total measure w. This measure transfers in the obvious way to curves transverse to the singular foliation.

This describes maps

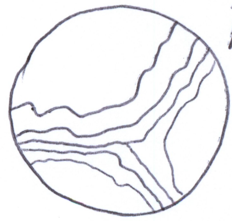
$$\left\{\begin{array}{c}\text{measured}\\\text{laminations}\end{array}\right\} \longrightarrow \left\{\begin{array}{c}\text{weighted}\\\text{train}\\\text{tracks}\end{array}\right\} \longrightarrow \left\{\begin{array}{c}\text{measured}\\\text{singular}\\\text{foliations}\end{array}\right\}.$$

We'll complete the picture with a map

$$\left\{\begin{matrix}\text{measured}\\ \text{singular}\\ \text{foliations}\end{matrix}\right\} \longrightarrow \left\{\begin{matrix}\text{measured}\\ \text{laminations}\end{matrix}\right\}.$$

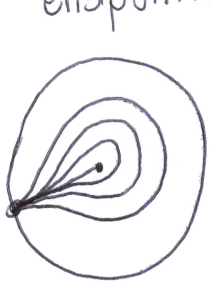
Consider a measured singular foliation $\mathcal{F}$ on M.
Lift $\mathcal{F}$ to a measured singular foliation $\tilde{\mathcal{F}}$ on $\tilde{M}$.
The boundary at infinity $\partial_\infty \tilde{M}$ is S^1.
Each smooth leaf of $\tilde{\mathcal{F}}$ lifts to a curve in $\tilde{M}$ with endpoints in $\partial_\infty \tilde{M}$.

$\tilde{M}$



Claim: The endpoints of a smooth leaf cannot coincide.

Pf:



If so there must be a "dead end" singular leaf, as shown.
This is not allowed. $\boxtimes$

So we can pull each smooth leaf in $\tilde{M}$ tight to a geodesic with the same endpoints.

FACT: Distinct leaves pull tight to disjoint geodesics.

This defines a π_1-equivariant map $\text{tight}: \begin{matrix}\text{smooth}\\ \text{leaves}\end{matrix} \to \begin{matrix}\text{geods}\\ \text{in } \tilde{M}\end{matrix}$.

The image of $\text{tight}(\mathcal{F})$ is a π_1-invariant geodesic lamination λ of $\tilde{M}$. For arc a transverse to λ define the measure $\mu(a)$ as the measure of

$\text{tight}^{-1}(a \cap \lambda)$. This defines a π_1-invariant measured lamination on $\tilde{M}$ that descends to a measured lamination on M.