Lecture 1 at HUJI 2009-03-18

(Notes from a course on mapping class groups by Peter Storm at Hebrew University.)

Fix a compact oriented surface M without boundary. Assume all map preserve orientation unless explicitly stated otherwise. Assume M is not a sphere.

Def: $\begin{pmatrix}\text{Mapping class}\\ \text{group of } M\end{pmatrix} = \mathrm{Mod}(M) := \pi_0(\mathrm{Homeo}(M))$

We will call elements of $\mathrm{Mod}(M)$ mapping classes. (orientation preserving homeos)

Thm (Dehn - Nielsen - Baer): The natural map

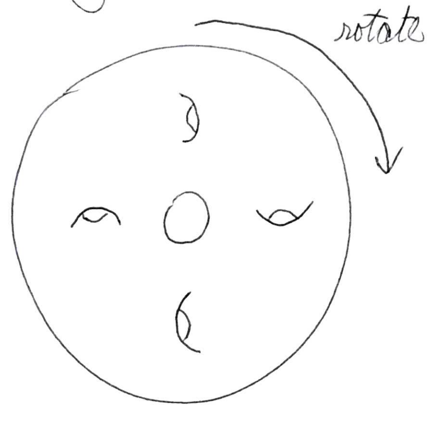
$\mathrm{Homeo}(M) \longrightarrow \mathrm{Out}(\pi_1 M)$ defines a homom.

$\mathrm{Mod}(M) \to \mathrm{Out}(\pi_1 M)$. This homom is injective with image of index 2. The image is exactly orientation preserving elts of $\mathrm{Out}(\pi_1 M)$, i.e. those acting trivially on $H^2(\pi_1 M; \mathbb{Z}) \cong \mathbb{Z}$.

E.g. If M is a torus then $\mathrm{Mod}(M) \cong Sl_2\mathbb{Z}$.

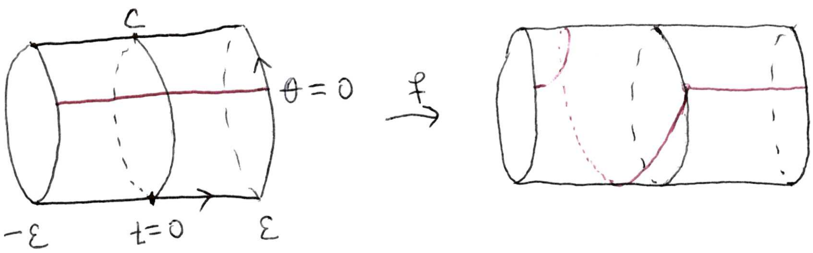
An example of a mapping class is the finite order homeom:

rotate by $\pi/2$



Another (more important) example is given by the Dehn twist. Let $c \subset M$ be an (embedded) homotopically nontrivial simple closed curve. A right hand Dehn twist about c, D_c, is a homeom $M \to M$ with support in a small annular ~~a~~ neigh. of c.

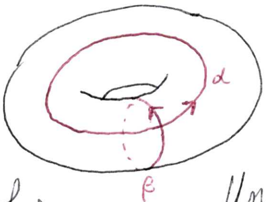
Put "coordinates" (t,θ) on an annular neigh. of c s.t. $t \in (-\varepsilon, \varepsilon)$, ~~and~~ $\theta \in S^1$, and $c = \{(0,\theta) \mid \theta \in S^1\}$.



Then define a homeo $f: M \to M$ with support in this annular nbd by

$$f(t,\theta) := \begin{cases} (t, e^{\frac{t}{\varepsilon}\cdot 2\pi i}) & \text{for } t \le 0 \\ (t,\theta) & \text{for } t > 0 \end{cases}$$

Then D_c is the mapping class of f. D_c is also called the left ~~-handed~~ Dehn twist about c. (I chose the left-handed orientation to follow N. Ivanov.) Note the def'n of f involved an orientation of c, namely which is the left side of c, but D_c is independent of this choice. Obviously, ~~with~~ we could have chosen f to be smooth with slightly more work.

On a torus 

, Dehn twists take a particularly simple linear form. Under the Dehn–Nielsen–Baer ~~monomorph~~ homom. D_β corresponds to

$$\begin{Bmatrix} \beta \longmapsto \beta \\ \alpha \longmapsto \alpha+\beta \end{Bmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

and D_α corresponds to

$$\begin{Bmatrix} \alpha \longmapsto \alpha \\ \beta \longmapsto \beta-\alpha \end{Bmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}.$$

Here Dehn twists correspond to unipotent elements. Intuitively, Dehn twists are similar to unipotents in higher genera.

In lattices, unipotents are not "typical" elements. This is true here also. Without defining "typical" precisely, I claim Dehn twists are not typical elements of $\mathrm{Mod}(M)$. What is an example of a typical element? Consider $\varphi = D_\alpha^{-1} D_\beta$, which corresponds to $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$, a hyperbolic element of $SL_2\mathbb{Z}$. I claim φ is typical. In particular, φ has the following property:

[Def: Let $\mathcal{S}$ be ~~a~~ the set of isotopy classes of homotopically nontrivial unoriented simple closed curves on M.

Clearly $\mathrm{Mod}(M) \curvearrowright \mathcal{S}$.

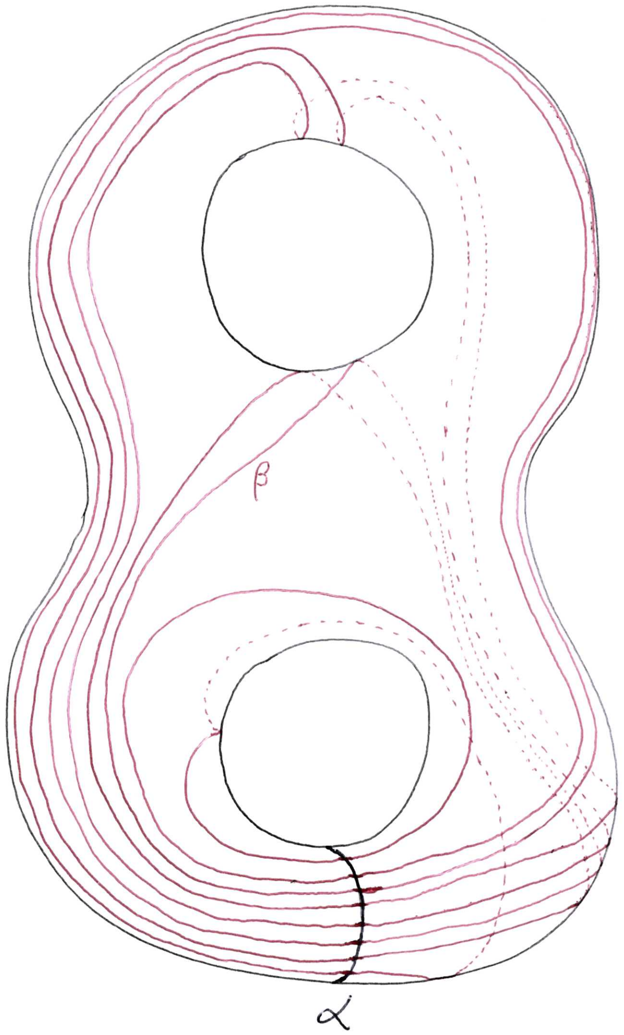
(*) [Then for all $k \neq 0$, φ^k acts on $\mathcal{S}$ without fixed pts., i.e. if c is a nontrivial s.c.c. then $\varphi^k(c)$ is not homotopic to c.]

We'll call this property (*). In this case it is easy to prove using linear algebra. For , $\mathcal{S}$ is $\left\{ \pm(m,n) \;\middle|\; \begin{array}{l} \text{If } m=0 \text{ then } n\neq 0. \\ \text{If } n=0 \text{ then } m \neq 0. \\ \text{Otherwise } \gcd(m,n)=1 \end{array} \right\}$.

How can be build higher genus mapping classes wish property (*)? Suppose α and β are simple closed curves on M satisfying

- they are homotopically nontrivial
- $M - (\alpha \cup \beta)$ is a set of disks
- If $\alpha' \sim \alpha$ and $\beta' \sim \beta$ then $\#|\alpha \cap \beta| \leq \#|\alpha' \cap \beta'|$, i.e. α & β intersect minimally.

For example if M has genus 2 then one could choose α & β as:



Then I claim that $D_\alpha^{-1} D_\beta$ has property (★) and should be considered a typical element of $\mathrm{Mod}(M)$. See (FLP, Ex. 13).

Measured singular foliations

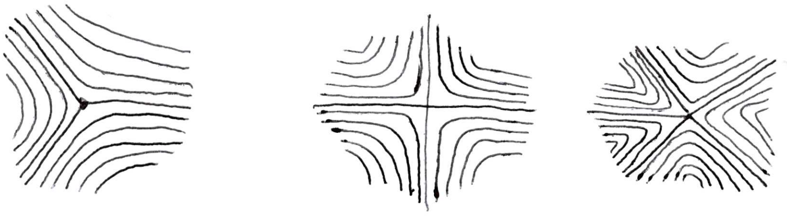
A measured singular foliation $\mathcal{F}$ is:

- a finite set $S = \{p_1, p_2, \ldots, p_N\} \subset M$ (the "singular set")
- an atlas $\{U_\alpha, \varphi_\alpha\}$ on $M - S$ such that each $U_\alpha = (a_1, b_1) \times (a_2, b_2)$, each transition $\varphi_\beta^{-1} \circ \varphi_\alpha$ takes horizontal lines to horizontal lines, and each transition $\varphi_\beta^{-1} \circ \varphi_\alpha$ preserves vertical distances. More specifically, these conditions say:

$$\left(\begin{array}{l} \text{If } (\varphi_\beta^{-1} \circ \varphi_\alpha)(x_i, y_i) = (x_i', y_i') \text{ for } i \in \{1,2\} \text{ then} \\ \qquad |y_1 - y_2| = |y_1' - y_2'|. \end{array} \right)$$

This atlas defines a horizontal foliation on $M - S$.

- Each singular point p_i has a neigh. such that the horizontal foliation looks like one of

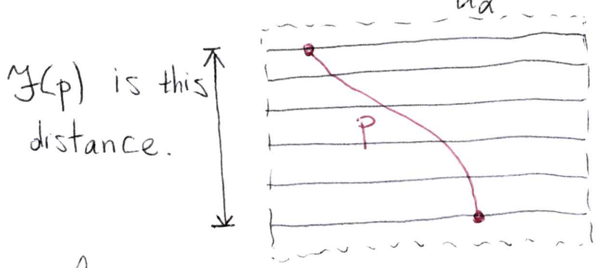


, ... (allow higher order singularities)

$\mathcal{F}$ defines a (positive) measure on the set of arcs of $M-S$ transverse to the horizontal foliation. Namely, if such an arc $p:[0,1]\to M-S$ is contained in a single chart then

$$\mathcal{F}(p) = \left| (y\text{-coord. of } p(1)) - (y\text{-coord of } p(0)\right|.$$

In gen'l, cut the arc into small pieces and sum.



Note that if $\chi(M)<0$ then the singular set ~~must be~~ S must be nonempty.

We will say two foliations $\mathcal{F}$ & $\mathcal{F}'$ are isotopic if there is a diffeo f on M isotopic to the identity such that:

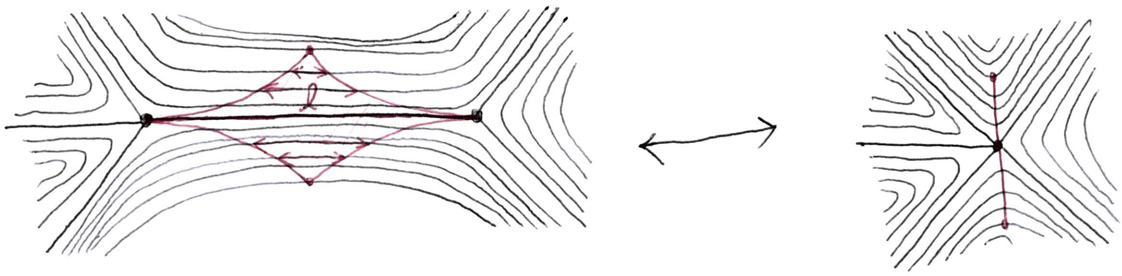
- $f(S) = f(S')$ ← singular set of $\mathcal{F}'$

- f sends the horiz. foliation of $\mathcal{F}$ to that of $\mathcal{F}'$.

- If p is an arc in $M-S$ transverse to $\mathcal{F}$ then
$$\mathcal{F}'(f(p)) = \mathcal{F}(p).$$

E.g. One could compose the charts with a small diffeo of M.

Next we'll describe Whitehead moves. Suppose l is a leaf of $\mathcal{F}$ running from one singular pt to another as shown.



In a nbd. of l cut out an open diamond shaped region as shown, and glue the endpts of horiz. leaves together. This produces the measured singular foliation on the right. This process is reversible. Both are called Whitehead moves. The two measured singular foliations are Whitehead equivalent.

Let A be the set of measured singular foliations on M.

Define

$$\mathcal{F}_1 \sim_1 \mathcal{F}_2 \iff \left\{ \begin{array}{l} \exists \text{ a sequence of Whitehead} \\ \text{moves on } \mathcal{F}_1 \text{ producing a measured} \\ \text{sing. foliation isotopic to } \mathcal{F}_2 \end{array} \right\}$$

and

$$\mathcal{F}_1 \sim_2 \mathcal{F}_2 \iff \left\{ \begin{array}{l} \exists\ \mathcal{F}_3 \text{ s.t. } \mathcal{F}_1 \sim_1 \mathcal{F}_3 \text{ and the} \\ \text{measure on } \mathcal{F}_3 \text{ is a constant multiple} \\ \text{of the measure on } \mathcal{F}_2 \end{array} \right\}.$$

Then $\mathcal{MF} := A/\sim_1$ and $\mathbb{P}\mathcal{MF} := A/\sim_2$.

Def: $\mathcal{F}^u$ & $\mathcal{F}^s$ foliations on M are transverse if they have the same singular set S and are transverse on $M - S$.

Def: $f: M \to M$ a homeomorphism is pseudo-Anosov (Ψ-A) if $\exists$ $\mathcal{F}^u$ & $\mathcal{F}^s$ transverse singular measured foliations and $\lambda > 1$ s.t.

- f sends leaves of $\mathcal{F}^*$ to leaves of $\mathcal{F}^*$
- $f(\mathcal{F}^s) = \frac{1}{\lambda} \mathcal{F}^s$ and $f(\mathcal{F}^u) = \lambda \mathcal{F}^u$,

  where these are equalities of measures.

Def: A multicurve $c \subset M$ is an embedded 1-manifold (i.e. a finite union of simple closed curves) s.t. each component is homotopically nontrivial, and no pair of components are isotopic.

Thurston's classification of surface homeos: A homeom $f: M \to M$ is isotopic to a homeo g satisfying at least one of the following:

(i) g has finite order in Homeo(M)

(ii) $\exists$ multicurve $c \subset M$ s.t. $g(c) = c$.

(iii) g is pseudo-Anosov.

(If (iii) then not (i) and not (ii).)