Some solutions to Homework 4

Pg 48, #3) The claim is equivalent to the following:

$$\sup P \le \sup Q$$

$\Leftrightarrow \forall x \in P : (x \le \sup Q)$ (sup Q upper bound of P)

$\Leftarrow \forall x \in Q : (x \le \sup Q)$ (Since $P \subseteq Q$)

The last condition is true by definition of sup.

#4) For simplicity, assume that $A, B \neq \emptyset$. Lets show that $\inf(A+B) \le \inf(A) + \inf(B)$ and $\inf(A+B) \ge \inf(A) + \inf(B)$:

1) $\inf(A+B) \ge \inf(A) + \inf(B)$

$\Leftrightarrow$ $\inf(A) + \inf(B)$ is a lower bound of $\inf(A+B)$

$\Leftrightarrow \forall z \in A+B : (z \ge \inf(A) + \inf(B))$

$\Leftrightarrow \forall x \in A, y \in B : (x+y \ge \inf(A) + \inf(B))$

$\Leftarrow$

$\forall x \in A, y \in B : (x \ge \inf(A)$ and $y \ge \inf(B))$, which is true by definition.

2) $\inf(A+B) \le \inf(A) + \inf(B)$

$\Leftrightarrow \forall \varepsilon > 0 \; \exists x \in A, y \in B : (x+y < \inf(A) + \inf(B) + \varepsilon)$

$\Leftarrow \forall \varepsilon > 0 \; \exists x \in A, y \in B : (x < \inf(A) + \frac{\varepsilon}{2}$ and $y < \inf(B) + \frac{\varepsilon}{2})$,

which is true because $\frac{\varepsilon}{2} > 0$.

Thus $\inf(A+B) = \inf(A) + \inf(B)$.

Pg 98 #8) Suppose $A, B \subseteq \mathbb{R}$, $A, B \neq \emptyset$.

a) True: $\sup(A \cap B) \leq \inf\{\sup A, \sup B\}$

$\Leftrightarrow \sup(A\cap B) \leq \sup(A)$ and $\sup(A \cap B) \leq \sup(B)$

$\Leftarrow A \cap B \subseteq A$ and $A \cap B \subseteq B$ (by Pg 48 # 3)

b) Not true: e.g. $A \cap B$ might be empty

c) and d) Both are true: let $t \in \mathbb{R}$ be arbitrary: then

$t \geq \sup\{\sup(A), \sup(B)\}$

$\Leftrightarrow$

$t \geq \sup(A)$ and $t \geq \sup(B)$

$\Leftrightarrow$

$\forall x \in A: (t \geq x)$ and ~~$\forall x$~~ $\forall x \in B: (t \geq x)$

$\Leftrightarrow \forall x \in A \cup B: (t \geq x)$

$\Leftrightarrow$ $t \geq \sup(A \cup B)$

Since this equivalence is true for all $t \in \mathbb{R}$, we get $\sup\{\sup(A), \sup(B)\} = \sup(A \cup B)$.

#15) Denote $a = d(x_0, x_1)$. Since we have $d(x_n, x_{n+1}) \leq \frac{1}{2} d(x_{n-1}, x_n)$ for all n, we get $d(x_n, x_{n+1}) \leq (\frac{1}{2})^n a$ for all $n \in \mathbb{N}$. If $N, n, m \in \mathbb{N}$ satisfy ~~$n, m \geq N$~~ $n \geq m \geq N$, then $d(x_n, x_m) \leq d(x_m, x_{m+1}) + \cdots + d(x_{n-1}, x_n) =$

$$= \sum_{k=m}^{n-1} d(x_k, x_{k+1}) \leq \sum_{k=m}^{n-1} \left(\tfrac{1}{2}\right)^k a \leq \sum_{k=m}^{\infty} \left(\tfrac{1}{2}\right)^k a = \frac{(\frac{1}{2})^m a}{1 - \frac{1}{2}} = \left(\tfrac{1}{2}\right)^{m-1} a \leq$$

$$\leq \left(\tfrac{1}{2}\right)^{N-1} a$$

If $\varepsilon > 0$, we can choose N large enough that $(\frac{1}{2})^{N-1} a < \varepsilon$.

Thus (x_n) is a Cauchy sequence.