Some solutions to Homework 3

Problem 1) Suppose that $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$.
Since every convergent sequence is bounded, we know there is $M>0$ s.t. $|x_n| \le M$ and $|y_n| \le M$ for all n.
This means that also $|x|, |y| \le M$.
Now we can see:

$$|x_n y_n - xy| = |x_n y_n - x_n y + x_n y - xy| \le |x_n|\cdot|y_n - y| + |y|\cdot|x_n - x|$$
$$\le M\cdot|y_n - y| + M\cdot|x_n - x|.$$

Let $\varepsilon>0$.
Choose N large enough such that $|x_n - x| < \frac{1}{2M}\varepsilon$ and $|y_n - y| < \frac{1}{2M}\varepsilon$ whenever $n \ge N$. Then for all $n \ge N$ we have

$$|x_n y_n - xy| \le M\cdot\frac{1}{2M}\varepsilon + M\cdot\frac{1}{2M}\varepsilon = \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \text{ so } \lim_{n\to\infty} x_n y_n = xy.$$

Pg 45 #3) Note that $x_n = \sqrt{n^2+1} - \sqrt{n^2} \ge 0$ and $x_n = \dfrac{1}{\sqrt{n^2+1}+n} \le \dfrac{1}{n}$ $\Big\}$ By Sandwich thm $\Rightarrow \lim_{n\to\infty} x_n = 0$

#4) We haven't really defined logarithm yet but $x_n = \ln(n)$ satisfies
$x_{n+1} - x_n \le \frac{1}{n}$ and $x_n \to \infty$ as $n \to \infty$.

#5) Suppose $\mathbb{F}$ satisfies the given condition and (x_n) is a (bounded!) monotone increasing sequence. We know that $y_n = \frac{-1}{n}$ is a <u>strictly</u> increasing sequence, bounded above by 0. It must then have a limit, call it y. Also (x_n+y_n) is a strictly increasing sequence, bounded above. Thus it has a limit, call it z.

This implies that $x_n = (x_n+y_n) - y_n$ converges to $y-z$, so $\mathbb{F}$ is complete.

Pg 97 #3) a) If $x \neq 0$, then $x > 0$. Pick $\varepsilon = \frac{x}{2} > 0$. Then we should have $x \leq \frac{x}{2}$, so $x - \frac{x}{2} = \frac{x}{2} \leq 0$, so $x \leq 0$. This is a contradiction, so $x = 0$.

b) If $\varepsilon > 0$, then $\frac{\varepsilon}{2} < \varepsilon$. Pick $x = \min\{\frac{1}{2}, \frac{\varepsilon}{2}\}$. Then $x < \varepsilon$ and $x \in S$.

#5) Lets prove that $\lim_{n\to\infty} x_n = \sup(S)$. Let $\varepsilon > 0$. We want to find N such that for all $n \geq N$, $\sup(S) - \varepsilon \leq x_n < \sup(S) + \varepsilon$.

By definition of $\sup(S)$, we always have $x_n \leq \sup(S)$, and there exists N such that $x_N > \sup(S) - \varepsilon$. Putting this together we get for $n \geq N$:

$$\sup(S) - \varepsilon < x_N \underset{(x_n)\text{ increasing}}{\leq} x_n \leq \sup(S) < \sup(S) + \varepsilon$$

Thus $\lim_{n\to\infty} x_n = \sup(S)$.

#25 a) Suppose $P \leq Q$. The claim $\sup P \leq \sup Q$ is equivalent to saying that $\sup Q$ is an upper bound of P.

If $x \in P$, then $\exists y \in Q$ s.t. $x \leq y$, but then also $y \leq \sup(Q)$ so $x \leq \sup Q$. Thus $\sup(Q)$ is an upper bound for P, so $\sup(Q) \geq \sup(P)$.