

# A Tridiagonal Matrix

We investigate the simple  $n \times n$  real tridiagonal matrix:

$$M = \begin{pmatrix} \alpha & \beta & 0 & 0 & \dots & 0 & 0 \\ \beta & \alpha & \beta & 0 & \dots & 0 & 0 \\ 0 & \beta & \alpha & \beta & \dots & 0 & 0 \\ \vdots & \vdots & & \ddots & & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \alpha & \beta & 0 \\ 0 & 0 & 0 & \dots & \beta & \alpha & \beta \\ 0 & 0 & 0 & \dots & 0 & \beta & \alpha \end{pmatrix} = \alpha I + \beta \begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & & \ddots & & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 1 & 0 \\ 0 & 0 & 0 & \dots & 1 & 0 & 1 \\ 0 & 0 & 0 & \dots & 0 & 1 & 0 \end{pmatrix} = \alpha I + \beta T,$$

where  $T$  is defined by the preceding formula. This matrix arises in many applications, such as  $n$  coupled harmonic oscillators and solving the Laplace equation numerically. Clearly  $M$  and  $T$  have the same eigenvectors and their respective eigenvalues are related by  $\mu = \alpha + \beta\lambda$ . Thus, to understand  $M$  it is sufficient to work with the simpler matrix  $T$ .

## Eigenvalues and Eigenvectors of $T$

Usually one first finds the eigenvalues and then the eigenvectors of a matrix. For  $T$ , it is a bit simpler first to find the eigenvectors. Let  $\lambda$  be an eigenvalue (necessarily real) and  $V = (v_1, v_2, \dots, v_n)$  be a corresponding eigenvector. With hindsight it will be convenient to write  $\lambda = 2c$ . Then

$$0 = (T - \lambda I)V = \begin{pmatrix} -2c & 1 & 0 & 0 & \dots & 0 & 0 \\ 1 & -2c & 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & -2c & 1 & \dots & 0 & 0 \\ \vdots & \vdots & & \ddots & & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -2c & 1 & 0 \\ 0 & 0 & 0 & \dots & 1 & -2c & 1 \\ 0 & 0 & 0 & \dots & 0 & 1 & -2c \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_{n-2} \\ v_{n-1} \\ v_n \end{pmatrix} \tag{1}$$

$$= \begin{pmatrix} -2cv_1 + v_2 \\ v_1 - 2cv_2 + v_3 \\ \vdots \\ v_{k-1} - 2cv_k + v_{k+1} \\ \vdots \\ v_{n-2} - 2cv_{n-1} + v_n \\ v_{n-1} - 2cv_n \end{pmatrix}$$

Except for the first and last equation, these have the form

$$v_{k-1} - 2cv_k + v_{k+1} = 0. \tag{2}$$

We can also bring the first and last equations into this same form by introducing new artificial variables  $v_0$  and  $v_{n+1}$ , setting their values as zero:  $v_0 = 0$ ,  $v_{n+1} = 0$ .

The result (2) is a *second order linear difference equation with constant coefficients* along with the *boundary conditions*  $v_0 = 0$ , and  $v_{n+1} = 0$ . As usual for such equations one seeks a solution with the form  $v_k = r^k$ . Equation (2) then gives  $1 - 2cr + r^2 = 0$  whose roots are

$$r_{\pm} = c \pm \sqrt{c^2 - 1}$$

Note also

$$2c = r + r^{-1} \quad \text{and} \quad r_+ r_- = 1. \quad (3)$$

**Case 1:**  $c \neq \pm 1$ . In this case the two roots  $r_{\pm}$  are distinct. Let  $r := r_+ = c + \sqrt{c^2 - 1}$ . Since  $r_- = c - \sqrt{c^2 - 1} = 1/r$ , we deduce that the general solution of (1) is

$$v_k = Ar_+^k + Br_-^k = Ar^k + Br^{-k}, \quad k = 0, \dots, n+1 \quad (4)$$

for some constants  $A$  and  $B$ .

The first boundary condition,  $v_0 = 0$ , gives  $A + B = 0$ , so

$$v_k = A(r^k - r^{-k}), \quad k = 0, \dots, n+1. \quad (5)$$

Since for a non-trivial solution we need  $A \neq 0$ , the second boundary condition,  $v_{n+1} = 0$ , implies

$$r^{n+1} - r^{-(n+1)} = 0, \quad \text{so} \quad r^{2(n+1)} = 1.$$

In particular,  $|r| = 1$ . Using (3), this gives  $2|c| \leq |r| + |r|^{-1} = 2$ . Thus  $|c| \leq 1$ . In fact,  $|c| < 1$  because we are assuming that  $c \neq \pm 1$ .

**Case 2:**  $c = \pm 1$ . Then  $r = c$  and the general solution of (1) is now

$$v_k = (A + Bk)c^k.$$

The boundary condition  $v_0 = 0$  implies that  $A = 0$ . The other boundary condition then gives  $0 = v_{n+1} = B(n+1)c^{n+1}$ . This is satisfied only in the trivial case  $B = 0$ . Consequently the equations (1) have no non-trivial solution for  $c = \pm 1$ .

It remains to rewrite our results in a simpler way. We are in Case 1 so  $|r| = 1$ . Thus  $r = e^{i\theta}$ ,  $c = \cos\theta$ , and  $1 = r^{2(n+1)} = e^{2i(n+1)\theta}$ . Consequently  $2(n+1)\theta = 2k\pi$  for some  $1 \leq k \leq n$  (we exclude  $k = 0$  and  $k = n+1$  because we know that  $c \neq \pm 1$ , so  $r \neq \pm 1$ ). Normalizing the eigenvectors  $V$  by the choice  $A = 1/2i$ , we summarize as follows:

**Theorem 1** The  $n \times n$  matrix  $T$  has the eigenvalues

$$\lambda_k = 2c = 2 \cos \theta = 2 \cos \frac{k\pi}{n+1}, \quad 1 \leq k \leq n$$

and corresponding eigenvectors

$$V_k = \left( \sin \frac{k\pi}{n+1}, \sin \frac{2k\pi}{n+1}, \dots, \sin \frac{nk\pi}{n+1} \right).$$

REMARK 1. If  $n = 2k + 1$  is odd, then the middle eigenvalue is zero because  $(k + 1)\pi / (n + 1) = (k + 1)\pi / 2(k + 1) = \pi / 2$ .

REMARK 2. Since  $2ab = a^2 + b^2 - (a - b)^2 \leq a^2 + b^2$  with equality only if  $a = b$ , we see that for any  $x \in \mathbb{R}^n$

$$\langle x, Tx \rangle = 2(x_1x_2 + x_2x_3 + \dots + x_{n-1}x_n) \leq x_1^2 + 2(x_2^2 + \dots + x_{n-1}^2) + x_n^2 \leq 2\|x\|^2$$

with equality only if  $x = 0$ . Similarly  $\langle x, Tx \rangle \geq -2\|x\|^2$ . Thus, the eigenvalues of  $T$  are in the interval  $-2 < \lambda < 2$ . Although we obtained more precise information above, it is useful to observe that we could have deduced this so easily.

REMARK 3. *Gershgorin's circle theorem* is also a simple way to get information about the eigenvalues of a square (complex) matrix  $A = (a_{ij})$ . Let  $D_i$  be the disk in the complex plane whose center is at  $a_{ii}$  and radius is  $R_i = \sum_{j \neq i} |a_{ij}|$ , so

$$|\lambda - a_{jj}| \leq R_j.$$

These are the *Gershgorin disks*.

**Theorem 2 (Gershgorin)** Each eigenvalues of  $A$  lies in at least one of these Gershgorin discs.

**Proof:** Say  $Ax = \lambda x$  and say  $|x_i| = \max_j |x_j|$ . The  $i^{\text{th}}$  component of  $Ax = \lambda x$  is

$$(\lambda - a_{ii})x_i = \sum_{j \neq i} a_{ij}x_j$$

so

$$|(\lambda - a_{ii})x_i| \leq \sum_{j \neq i} |a_{ij}||x_j| \leq R_i|x_i|.$$

That is,  $|\lambda - a_{ii}| \leq R_i$ , as claimed.

**Example** By Gershgorin's theorem, we observed immediately that all of the eigenvalues of  $T$  satisfy  $|\lambda| \leq 2$ .

## Determinant of $T - \lambda I$

We use recursion on  $n$ , the size of the  $n \times n$  matrix  $T$ . It will be convenient to build on (1) and let  $D_n = \det(T - \lambda I)$ . As before, write  $\lambda = 2c$ . Then, expanding by minors using the first column of (1) we obtain the formula

$$D_n = -2cD_{n-1} - D_{n-2} \quad n = 3, 4, \dots \quad (6)$$

Since  $D_1 = -2c$  and  $D_2 = 4c^2 - 1$ , we can use (6) to define  $D_0 := 1$ . The relation (6) is, except for the sign of  $c$ , is identical to (2). The solution for  $c \neq \pm 1$  is thus

$$D_k = As^k + Bs^{-k}, \quad k = 0, 1, \dots, \quad (7)$$

where

$$-2c = s + s^{-1} \quad \text{and} \quad s = -c + \sqrt{c^2 - 1}. \quad (8)$$

This time we determine the constants  $A, B$  from the *initial conditions*  $D_0 = 1$  and  $D_1 = -2c$ . The result is

$$D_k = \begin{cases} \frac{1}{2\sqrt{c^2 - 1}}(s^{k+1} - s^{-(k+1)}) & \text{if } c \neq \pm 1, \\ (-c)^k(k+1) & \text{if } c = \pm 1. \end{cases} \quad (9)$$

For many purposes it is useful to rewrite this.

**Case 1:**  $|c| < 1$ . Then  $s = -c + i\sqrt{1 - c^2}$  has  $|s| = 1$  so  $s = e^{i\alpha}$  and  $c = -\cos \alpha$  for some  $0 < \alpha < \pi$ . Therefore from (9),

$$D_k = \frac{\sin(k+1)\alpha}{\sin \alpha}. \quad (10)$$

**Case 2:**  $c > 1$ . Write  $c = \cosh \beta$  for some  $\beta > 0$ . Since  $-e^\beta - e^{-\beta} = -2c = s + s^{-1}$ , write  $s = -e^\beta$ . Then from (9),

$$D_k = (-1)^k \frac{\sinh(k+1)\beta}{\sinh \beta}, \quad (11)$$

where we chose the sign in  $\sqrt{c^2 - 1} = -\sinh \beta$  so that  $D_0 = 1$ .

**Case 3:**  $c < -1$ . Write  $c = -\cosh \beta$  for some  $\beta > 0$ . Since  $e^\beta + e^{-\beta} = -2c = s + s^{-1}$ , write  $s = e^\beta$ . Then from (9),

$$D_k = \frac{\sinh(k+1)\beta}{\sinh \beta}, \quad (12)$$

where we chose the sign in  $\sqrt{c^2 - 1} = +\sinh \beta$  so that  $D_0 = 1$ .

Note that as  $t \rightarrow 0$  in (10)–(12), that is, as  $c \rightarrow \pm 1$ , these formulas agree with the case  $c = \pm 1$  in (9).

## A Vibrating String (coupled oscillators)

Say we have  $n$  particles with the same mass  $m$  equally spaced on a string having tension  $\tau$ . Let  $y_k$  denote the vertical displacement of the  $k^{\text{th}}$  mass. Assume the ends of the string are fixed; this is the same as having additional particles at the ends, but with zero displacement:  $y_0 = 0$  and  $y_{n+1} = 0$ . Let  $\phi_k$  be the angle the segment of the string between the  $k^{\text{th}}$  and  $(k+1)^{\text{st}}$  particle makes with the horizontal. Then Newton's second law of motion applied to the  $k^{\text{th}}$  mass asserts that

$$m\ddot{y}_k = \tau \sin \phi_k - \tau \sin \phi_{k-1}, \quad k = 1, \dots, n. \quad (13)$$

If the particles have horizontal separation  $h$ , then  $\tan \phi_k = (y_{k+1} - y_k)/h$ . For the case of small vibrations we assume that  $\phi_k \approx 0$ ; then  $\sin \phi_k \approx \tan \phi_k = (y_{k+1} - y_k)/h$  so we can rewrite (13) as

$$\ddot{y}_k = p^2(y_{k+1} - 2y_k + y_{k-1}), \quad k = 1, \dots, n, \quad (14)$$

where  $p^2 = \tau/mh$ . This is a system of second order linear constant coefficient differential equations with the boundary conditions  $y_0(t) = 0$  and  $y_{n+1}(t) = 0$ . As usual, one seeks special solutions of the form  $y_k(t) = v_k e^{\alpha t}$ . Substituting this into (14) we find

$$\alpha^2 v_k = p^2(v_{k+1} - 2v_k + v_{k-1}), \quad k = 1, \dots, n,$$

that is,  $\alpha^2$  is an eigenvalue of  $p^2(T - 2I)$ . From the work above we conclude that

$$\alpha_k^2 = -2p^2(1 - \cos \frac{k\pi}{n+1}) = -4p^2 \sin^2 \frac{k\pi}{2(n+1)}, \quad k = 1, \dots, n,$$

so

$$\alpha_k = 2ip \sin \frac{k\pi}{2(n+1)}, \quad k = 1, \dots, n.$$

The corresponding eigenvectors  $V_k$  are the same as for  $T$ . Thus the special solutions are

$$Y_k(t) = V_k e^{2ipt \sin \frac{k\pi}{2(n+1)}}, \quad k = 1, \dots, n,$$

where  $Y(t) = (y_1(t), \dots, y_n(t))$ .