Homework Set 9

DUE: Thurs, April 9, 2009. Late papers accepted until 1:00 Friday.

1. Let $f(z)$ be holomorohic in the unit disk, $|z| < 1$ with $|f(z)| < M$ in the closed disk $|z| \le 1$. Say $f(c_k) = 0$, where $|c_k| \nearrow 1$. If $\sum(1-|c_k|) = \infty$, show that $f(z)$ is identically zero, while if $\sum(1-|c_k|) < \infty$, exhibit such a function that is not identically zero.

   [HINT: Use the Blaschke product $\varphi(z) = \prod_k \left(\frac{z-c_k}{1-\bar{c}_k z}\right)$.]

2. a) Give an example of a function $f(z)$ is holomorphic in $|z| \le 1$ and satisfies $|f(z)| = 1$ on $|z| = 1$, but f is not an entire function.

   b) Show that every such function must be rational.

3. Prove that the roots of the polynomian $p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_0$ depend continuously on the coefficients $a_0, \cdots, a_{n-1}$.

4. Show that the functions z^n, $n = 0, 1, 2, \ldots$ form a normal family in $|z| < 1$.

5. Assume the disk $A = \{|z-a| \le r\}$ is inside $D = \{|z| < 1\}$. Find a univalent conformal map of the annular region between A and D onto an annulus of the form $\{\rho < |w| < 1\}$.

   Equivalently, let C_1 and C_2 be nonintersecting circles in the complex plane. Show there is a Möbius transformation that maps them onto *concentric* circles.

6. The points z_1 and z_2 are *symmetric with respect to a circle* if every circle (or straight line) that contains these points intersects the circle orthogonally. If z_1 and z_2 are symmetric with respect to a circle C and $h(z)$ is a Möbius transformation, show that their images $w_j = h(z_j)$ are symmetric with respect to the image of the circle, $h(C)$.

7. Let $\Omega \in \mathbb{C}$ be the intersection of the disks $|z-1| < 2$ and $|z+1| < 2$.

   a) Find an injective conformal map from Ω to the unit disk.

   b) Is there a conformal map that maps points on the imaginary axis in Ω to the interval between $\pm i$ on the imaginary axis? If so, is this map uniquely determined?

8. Let $\Omega \in \mathbb{C}$ be a (connected) simply connected open set that is symmetric under the map $z \to \bar{z}$ (symmetric across the real axis).

   a) Can you always find a conformal map from Ω to the open unit disk that maps the real points in Ω to points on the real axis?

   b) Can you find a map f with the property that $f(\bar{z}) = \overline{f(z)}$?

9. A real function $u(x,y)$ is *biharmonic* if it satisfies $\Delta^2 u = 0$. Here

$$\Delta^2 u = \Delta(\Delta u) = u_{xxxx} + 2u_{xxyy} + u_{yyyy}.$$

Show that locally a biharmonic $u(x,y)$ has the form

$$u(x,y) = \mathrm{Re}\,[\bar{z}\varphi(z) + \psi(z)],$$

where φ and ψ are analytic.