Math 609 Jerry L. Kazdan
April 22, 2004 1:30-2:50

Exam 2

DIRECTIONS This exam has two parts, Part A is short answer (35 points) while Part B has traditional problems (60 points). All contour integrals are assumed to be in the positive sense (counterclockwise).

Short Answer Problems [5 points each] (35 points total)
For A1–A5 let $f(z)$ be holomorphic for $0 < |z| < \infty$. What can you say about $f(z)$ if you are told the following? Briefly justify your assertions.

A1. $|z^2 f(z)| < 5$.

A2. $|f(z)| \to \infty$ as $|z| \to 0$.

A3. $f(\frac{1}{n}) = 1 + (-1)^n, \qquad n = 1, 2, \ldots$.

A4. $|f(z)| \le |z| + 1$ and $f(\frac{1}{n}) = 0, \qquad n = 1, 2, \ldots$.

A5. $|f(z)| \le |f(3)|$ for $|z - 3| < 2$.

A6. Evaluate $\displaystyle \frac{1}{2\pi i} \oint_{|z-1|=2} \frac{e^{2z}}{z^2}\, dz$.

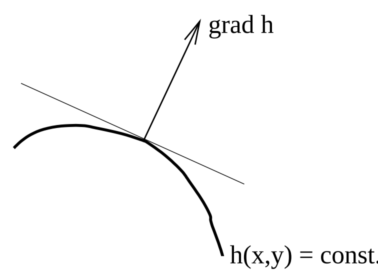
A7. Describe the singularities of $\varphi(z) := \dfrac{1 - \cos(z^5)}{\sin^3 z}$ at $z = 0$ and at $z = \pi$.

Traditional Problems [10 points each] (60 points total)

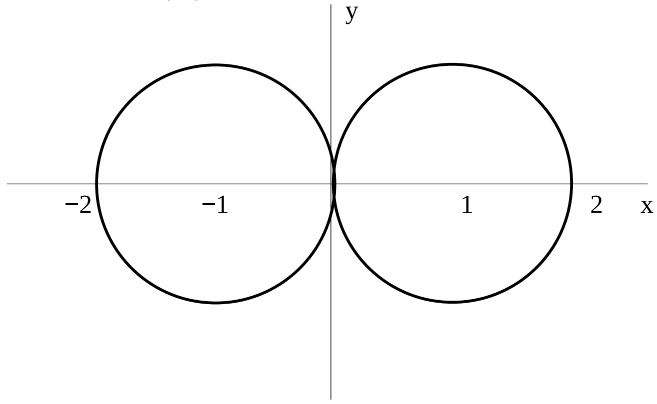
B1. Let $g(z)$ be holomorphic in the disk $\{|z| \le 3\}$ with $|g(z)| \le 7$ on the circle $\{|z| = 3\}$. Find some explicit upper bound for $|g'(z)|$ in the disk $\{|z| \le 1\}$.

B2. Let $f(z) = u + iv$ be holomorphic at $z_0 = x_0 + iy_0$ and $f'(z_0) \ne 0$. Show that the level curves of u and v through z_0 intersect orthogonally.

[You may use without (the simple) proof that if
i). $h(x, y) = \text{const}$ is a level curve of the smooth real-valued function $h(x, y)$ and
ii). the gradient $\nabla h(x_0, y_0) \ne 0$ at a point on this curve,
then $\nabla h(x_0, y_0)$ is orthogonal to the tangent line of h at (x_0, y_0).]



B3. Let $\Omega \in \mathbb{C}$ be the region *exterior* to the two disks $|z-1|<1$ and $|z+1|<1$. Find a conformal map $w=f(z)$ from Ω to the horizontal strip $-1<\text{Im}\{w\}<1$.



B4. Let $h(z)$, $z=x+iy$, be holomorphic in the strip $|y|<10$ with $|h(z)|<1$ there. Prove that $\cos z+h(z)$ has an infinite number of zeroes in this strip. [NOTE: $|\cos z|^2=\cosh^2 y-\sin^2 x$].

B5. For real λ let $I(\lambda):=\int_{-\infty}^{\infty} e^{-(x+i\lambda)^2}\,dx$. Show that $I(\lambda)=I(0)$ for all real λ.

SUGGESTION: Consider a contour integral around a rectangle with corners at $\pm R$ and $\pm R+i\lambda$.

[Remark: This is the main step in showing that $f(x):=\frac{1}{\sqrt{2\pi}}e^{-x^2/2}$ is its own Fourier transform.]

B6. Consider $\sum_{n=1}^{\infty}\frac{(-1)^n}{n!(n-z)}$. Let $K\subset\mathbb{C}$ be a compact set that does not contain any positive integers, $z=1,2,\ldots$. Show that the series converges uniformly on K to an analytic function.