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Partial Differential Equations

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CHAPTER 1

Introduction

Partial Differential Equations (PDEs) arise in many applications to physics, geometry, and more recently the world of finance. This will be a basic course.

In real life one can find explicit solutions of very few PDEs – and many of these are infinite series whose secrets are complicated to extract. For more than a century the goal is to understand the solutions – even though there may not be a formula for the solution.
The historic heart of the subject (and of this course) are the three fundamental linear equations: wave equation, heat equation, and Laplace equation along with a few nonlinear equations such as the minimal surface equation and others that arise from problems in the calculus of variations.

We seek insight and understanding rather than complicated formulas.

Prerequisites: Linear algebra, calculus of several variables, and basic ordinary differential equations. In particular I'll assume some experience with the Stokes' and divergence theorems and a bit of Fourier analysis. Previous acquaintantance with normed linear spaces will also be assumed. Some of these topics will be reviewed a bit as needed.

References: For this course, the most important among the following are the texts by Strauss and Evans.

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1. Functions of Several Variables

Partial differential equations work with functions of several variables, such as $u(x, y)$. Acquiring intuition about these can be considerably more complicated than functions of one variable. To test your intuition, here are a few questions concerning a smooth function $u(x, y)$ of the two variables x, y defined on all of $\mathbb{R}^2$.

Exercises:

1. Say $u(x, y)$ is a smooth function of two variables that has an isolated critical point at the origin (a *critical point* is where the gradient is zero). Say as you approach the origin along any straight line u has a local minimum. Must u have a local minimum if you approach the origin along any (smooth) curve? Proof or counter example.

2. There is no smooth function $u(x, y)$ that has exactly two isolated critical points, both of which are local local minima. Proof or counter example.

3. Construct a function $u(x, y)$ that has exactly three isolated critical points: one local max, one local min, and one saddle point.

4. A function $u(x, y)$, $(x, y) \in \mathbb{R}^2$ has exactly one critical point, say at the origin. Assume this critical point is a strict local minimum, so the second derivative matrix (or *Hessian matrix*).

$$u''(x, y) = \begin{pmatrix} u_{xx} & u_{xy} \\ u_{xy} & u_{yy} \end{pmatrix}$$

is positive definite at the origin. Must this function have its global minimum at the origin, that is, can one conclude that $u(x, y) > u(0, 0)$ for all $(x, y) \neq (0, 0)$?
Proof or counter example.

2. Classical Partial Differential Equations

Three models from classical physics are the source of most of our knowledge of partial differential equations:

$u_{tt} = u_{xx} + u_{yy}$	wave equation
$u_t = u_{xx} + u_{yy}$	heat equation
$u_{xx} + u_{yy} = f(x,y)$	Laplace equation

The homogeneous Laplace equation, $u_{xx} + u_{yy} = 0$, can be thought of as a special case of the wave and heat equation where the function $u(x,y,t)$ is independent of t. This course will focus on these equations. For all of these equations one tries to find explicit solutions, but this can be done only in the simplest situations. An important goal is to seek qualitative understanding, even if there are no useful formulas.

Wave Equation: Think of a solution $u(x,y,t)$ of the wave equation as describing the motion of a drum head Ω at the point (x,y) at time t. Typically one specifies

initial position: $u(x,y,0)$,
initial velocity: $u_t(x,y,0)$
boundary conditions: $u(x,y,t)$ for $(x,y) \in \partial\Omega, \quad t \ge 0$

and seek the solution $u(x,y,t)$.

Heat Equation: For the heat equation, $u(x,y,t)$ represents the temperature at (x,y) at time t. Here a typical problem is to specify

initial temperature: $u(x,y,0)$
boundary temperature: $u(x,y,t)$ for $(x,y) \in \partial\Omega$, $t \ge 0$

and seek $u(x,y,t)$ for $(x,y) \in \Omega$, $t > 0$. Note that if one investigates heat flow on the surface of a sphere or torus (or compact manifolds *without* boundary), then there are no boundary conditions for the simple reason that there is no boundary.

Laplace Equation: It is clear that if a solution $u(x,y,t)$ is *independent* of t, so one is in equilibrium, then u is a solution of the Laplace equation (these are called *harmonic functions*). Using the heat equation model, a typical problem is the *Dirichlet problem*, where one is given

$$\text{boundary temperature} \quad u(x,y,t) \qquad \text{for} \quad (x,y) \in \partial\Omega$$

and one seeks the (equilibrium) temperature distribution $u(x,y)$ for $(x,y) \in \Omega$. From this physical model, it is intuitively plausible that in equilibrium, the maximum (and minimum) temperatures can not occur at an interior point of Ω unless $u \equiv$ const., for if there were a local maximum temperature at an interior point of Ω, then the heat would

flow away from that point and contradict the assumed equilibrium. This is the *maximum principle*: if u satisfies the Laplace equation then

$$\min_{\partial\Omega} u \le u(x,y) \le \max_{\partial\Omega} u \qquad \text{for} \quad (x,y) \in \Omega.$$

Of course, one must give a genuine mathematical proof as a check that the differential equation really does embody the qualitative properties predicted by physical reasoning such as this.
For many mathematicians, a more familiar occurrence of harmonic functions is as the real or imaginary parts of analytic functions. Indeed, one should expect that harmonic functions have all of the properties of analytic functions — with the important exception that the product or composition of two harmonic functions is almost never harmonic (that the set of analytic functions is also closed under products, inverse (that is $1/f(z)$) and composition is a significant aspect of their special nature and importance).

Some Other Equations: It is easy to give examples of partial differential equations where little of interest is known. One example is the so-called *ultrahyperbolic* equation

$$u_{ww} + u_{xx} = u_{yy} + u_{zz}.$$

As far as I know, this does not arise in *any* applications, so it is difficult to guess any interesting phenomena; as a consequence it is of not much interest.
We also know little about the *local solvability* of the Monge-Ampère equation

$$u_{xx}u_{yy} - u_{xy}^2 = f(x,y)$$

near the origin in the particularly nasty case $f(0,0) = 0$, although at first glance it is not obvious that this case is difficult. This equation arises in both differential geometry and elasticity – and any results would be interesting to many people.

In partial differential equations, developing techniques are frequently more important than general theorems.
Partial differential equations, a nonlinear heat equation, played a central role in the recent proof of the Poincaré conjecture which concerns characterizing the sphere, S^3, topologically.
They also are key in the Black-Scholes model of how to value options in the stock market.

Our understanding of partial differential equations is rather primitive. There are fairly good results for equations that are similar to the wave, heat, and Laplace equations, but there is a vast wilderness, particularly for nonlinear equations.

3. Ordinary Differential Equations, a Review

Since some of the ideas in partial differential equations also appear in the simpler case of ordinary differential equations, it is important to grasp the essential ideas in this case.
We briefly discuss the main ODEs one can solve.

a). Separation of Variables. The equation $\dfrac{du}{dt} = f(t)g(u)$ is solved using separation of variables:

$$\frac{du}{g(u)} = f(t)dt.$$

Now integrate both sides and solve for u. While one can rarely explicitly compute the integrals, the view is that this is a victory and is as much as one can expect.
A special case is $\dfrac{du}{dt} + a(t)u = 0$, the *homogeneous first order linear equation*. Separation of variables gives

$$u(t) = e^{-\int^t a(x)\,dx}.$$

b). First Order Linear Inhomogeneous Equations. These have the form

$$\frac{du}{dt} + a(t)u = f(t).$$

When I first saw the complicated explicit formula for the solution of this, I thought it was particularly ugly:

$$u(t) = e^{-\int^t a(x)\,dx} \int^t f(x)\, e^{\int^x a(s)\,ds}\, dx$$

but this really is an illustration of a beautiful simple, important, and really useful general idea: try to transform a complicated problem into one that is much simpler. Find a function $p(t)$ so that the change of variable

$$u(t) = p(t)v(t)$$

reduces our equation to the much simpler

$$\frac{dv}{dt} = g(t), \tag{1.1}$$

which we solve by integrating both sides. Here are the details. Since

$$Lu = L(pv) = (pv)' + apv = pv' + (p' + ap)v,$$

if we pick p so that $p' + ap = 0$ then solving $Lu = f$ becomes $pv' = f$ which is just $Dv = (1/p)f$, where $Dv := v'$, as desired in (1.1). More abstractly, with this p define the operator $Sv := pv$ which multiplies v by p. The inverse operator is $S^{-1}w = (1/p)w$. The computation we just did says that for any function v

$$LSv = SDv, \quad \text{that is} \quad S^{-1}LS = D,$$

so using the change of variables defined by the operator S, the differential operator L is "similar" to the basic operator D. Consequently we can reduce problems concerning L to those for D.

Exercise: With $Lu := Du + au$ as above, we seek a solution $u(t)$, periodic with period 1 of $Lu = f$, assuming $a(t)$ and f are also periodic, $a(t+1) = a(t)$ etc. It will help to introduce the *inner product*

$$\langle g,\, h\rangle = \int_0^1 g(t)h(t)\,dt.$$

We say that g is *orthogonal* to h if $\langle g,\, h\rangle = 0$. Define the operator L^* by the rule $L^*w = -Dw + aw$.

a) Show that for all periodic u and w we have $\langle Lu,\, w\rangle = \langle u,\, L^*w\rangle$.
b) Show that for a given function f there is a periodic solution of $Lu = f$ if and only if f is orthogonal to all the (periodic) solutions z of the homogeneous equation $L^*z = 0$.

c). $\dfrac{d^2u}{dt^2} + c^2u = 0$, with $c \neq 0$ a constant. Before doing anything else, we can *rescale* the variable t, replacing t by t/c to reduce to the special case $c = 1$. Using scaling techniques can lead to deep results.
The operator $Lu := u'' + c^2u = 0$ has two types of invariance: i). *linearity* in u and *translation invariance* in t.
Linearity in u means that

$$L(u+v) = Lu + Lv, \quad \text{and} \quad L(au) = aLu$$

for any constant a.
To define translation invariance, introduce the simple *translation operator* T_α by

$$(T_\alpha u)(t) = u(t+\alpha)$$

Then L being translation invariant means that

$$L(T_\alpha u) = T_\alpha L(u) \tag{1.2}$$

for "any" function u. There is an obvious group theoretic property: $T_\alpha T_\beta = T_{\alpha+\beta}$.

Lemma [Uniqueness] *If $Lu = 0$ and $Lv = 0$ with both $u(0) = v(0)$ and $u'(0) = v'(0)$, then $u(t) = v(t)$ for all t.*
Proof: Let $w = u - v$. Introduce the "energy"

$$E(t) = \tfrac{1}{2}(w'^2 + w^2).$$

By linearity $w'' + w = 0$ so $E'(t) = w'(w'' + w) = 0$. This proves that $E(t)$ is a constant, that is, energy is conserved. But $w(0) = w'(0) = 0$ also implies that $E(0) = 0$, so $E(t) \equiv 0$. Consequently $w(t) = 0$ for all t, and hence $u(t) = v(t)$. □

We now use this. Since $\cos t$ and $\sin t$ are both solutions of $Lu = 0$, by linearity for any constants a and b the function $\phi(t) := a\cos t + b\sin t =$

0 is a solution of $L\phi(t) = 0$. By translation invariance, for any constant α, the function $z = \cos(t+\alpha)$ satisfies $L(z) = 0$. Claim: we can find constants a and b so that i) $z(0) = \phi(0)$ and ii) $z'(0) = \phi'(0)$. These two conditions just mean

$$\cos(\alpha) = a \quad \text{and} \quad -\sin(\alpha) = b.$$

Consequently, by the uniqueness lemma, we deduce the standard trigonometry formula

$$\cos(t+\alpha) = \cos\alpha\cos t - \sin\alpha\sin t.$$

Moral: one can write the general solution of $u'' + u = 0$ as either

$$u(t) = C\cos(t+\alpha)$$

for any constants C and α, or as

$$u(t) = a\cos t + b\sin t.$$

Physicists often prefer the first version which emphasises the time invariance, while mathematicians prefer the second that emphasizes the linearity of L.

Exercise: Consider solutions of the equation

$$Lu := u'' + b(t)u' + c(t)u = f(t),$$

where for some constant M we have $|\,b(t)\,| < M$ and $|\,c(t)\,| < M$. Generalize the uniqueness lemma. [SUGGESTION. Use the same $E(t)$ (which is an artificial substitute for "energy") but this time show that

$$E'(t) \le kE(t) \qquad \text{for some constant } k.$$

This means $[e^{-kt}E(t)]' \le 0$. Use this to deduce that $E(t) \le e^{kt}E(0)$ for all $t \ge 0$, so the energy can grow at most exponentially].

Exercise: If a map L is translation invariant [see (1.2)], and $q(t;\lambda) := Le^{\lambda t}$, show that $q(t;\lambda) = g(0;\lambda)e^{\lambda t}$. Thus, writing $Q(\lambda) = q(0;\lambda)$, conclude that

$$Le^{\lambda t} = Q(\lambda)e^{\lambda t},$$

that is, $e^{\lambda t}$ is an eigenfunction of L with eigenvalue $Q(\lambda)$. You find special solutions of the homogeneous equation by finding the values of λ where $Q(\lambda) = 0$.

Exercise: Use the previous exercise to discuss the second order linear difference equation $u(x+2) = u(x+1) + u(x)$. Then apply this to find the solution of

$$u(n+2) = u(n+1) + u(n), \quad n = 0, 1, 2, \ldots$$

with the initial conditions $u(0) = 1$, and $u(1) = 1$.

d). Group Invariance. One can use *group invariance* as the key to solving many problems. Here are some examples:

a) $au'' + bu' + cu = 0$, where a, b, and c are constants. This linear equation is also invariant under translation $t \mapsto t+\alpha$, as the example above. One seeks special solutions that incorporate the translation invariance and then use the linearity to build the general solution.

b) $at^2u'' + btu' + cu = 0$, where a, b, and c are constants. This is invariant under the similarity $t \mapsto \lambda t$. One seeks special solutions that incorporate the similarity invariance and then use the linearity to build the general solution.

c) $\dfrac{du}{dt} = \dfrac{at^2 + bu^2}{ct^2 + du^2}$, where a, b, c, and d are constants. This is invariant under the stretching

$$t \mapsto \lambda t, \quad u \mapsto \lambda u, \quad \text{for} \quad \lambda > 0.$$

In each case the idea is to seek a special solution that incorporates the invariance. For instance, in the last example, try $v(t) = \dfrac{u}{t}$.

Lie began his investigation of what we now call Lie Groups by trying to use Galois' group theoretic ideas to understand differential equations.

e). Local vs Global: nonlinear. . Most of the focus above was on local issues, say solving a differential equation $du/dt = f(t,u)$ for small t. A huge problem remains to understand the solutions for large t. This leads to the qualitative theory, and requires wonderful new ideas from topology. Note, however, that for nonlinear equations (or linear equations with singularities), a solution might only exist for finite t. The simplest example is

$$\frac{du}{dt} = u^2 \quad \text{with initial conditions} \quad u(0) = c.$$

The solution, obtained by separation of variables,

$$u(t) = \frac{c}{1-ct}$$

blows up at $t = 1/c$.

f). Local vs Global: boundary value problems. Global issues also arise if instead of solving an *initial value problem* one is solving a *boundary value problem* such as

(1.3)
$$\frac{d^2u}{dx^2} + a^2u = f(x) \quad \text{with boundary conditions} \quad u(0) = 0, \quad u(\pi) = 0.$$

Here one only cares about the interval $0 \le x \le \pi$. As the following exercise illustrates, even the case when a is a constant gives non-obvious results.

Exercise:

a) In the special case of (1.3) where $a = 0$, show that a solution exists for any f.

b) If $a = 1$, show that a solution exists if and only if $\int_0^\pi f(x)\sin x\,dx = 0$.

c) If $0 \le a < 1$ is a constant, show that a solution exists for any f.

Exercise: [Maximum Principle]

a) Let $u(x)$ be a solution of $-u'' + u = 0$ for $0 < x < 1$. Show that at a point $x = x_0$ where u has a local maximum, u cannot be positive. If $u(x_0) = 0$, what can you conclude?

b) Generalize to solutions of $-u'' + b(x)u' + c(x)u = 0$, assuming $c(x) > 0$.

c) Say u and v both satisfy $-u'' + u = f(x)$ for $0 < x < 1$ with $u(0) = v(0)$ and $u(1) = v(1)$. Show that $u(x) = v(x)$ for all $0 \le x \le 1$.

d) Say u is a periodic solution, so $u(1) = u(0)$ and $u'(1) = u'(0)$, of

$$-u'' = 1 - h(x)e^u \quad \text{for} \quad 0 \le x \le 1,$$

where h is also periodic and satisfies $0 < a \le h(x) \le b$. Find upper and lower bounds for u in terms of the constants a and b.

CHAPTER 2

First Order Linear Equations

1. Introduction

The local theory of a single first order partial differential equation, such as

$$2\frac{\partial u}{\partial x} - 3\frac{\partial u}{\partial y} = f(x, y),$$

is very special since everything reduces to solving ordinary differential equations. However the theory gets more interesting if one seeks a solution in some open set Ω or if one looks at a "global" problem.
We'll see some of the standard ideas here. Because the main basic ideas in studying partial differential equations arise more naturally when one investigates the wave, Laplace, and heat equations, we will not linger long on this chapter.
The story for a nonlinear equation, such as *Inviscid Burger's Equation*, $u_t + uu_x = 0$, is much more interesting. We may discuss it later.

2. The Equation $u_y = f(x, y)$

The simplest partial differential equation is surely

$$u_y(x, y) = f(x, y), \tag{2.1}$$

so given $f(x, y)$ one wants $u(x, y)$. This problem is not quite as trivial as one might think.

a). The homogeneous equation. If $\Omega \in \mathbb{R}^2$ is a disk, the most general solution of the homogeneous equation

$$u_y(x, y) = 0 \tag{2.2}$$

in Ω is

$$u(x, y) = \varphi(x), \tag{2.3}$$

for any function φ depending only on x.

The differential equation asserts that $u(x, y)$ is constant on the vertical lines. The vertical lines are called the *characteristics* of this differential equation. If Ω is a more complicated region (see figure), then the above result is not the most general solution since to the right of the y-axis one can use two different functions $\varphi_1(x)$ and $\varphi_2(x)$, one in each region. Thus, for simplicity we will restrict our attention to "vertically convex" domains Ω, that is, ones in which every vertical line intersects Ω in a single line segment.

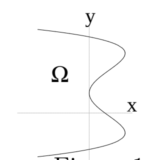


Figure 1-1

By analogy with ordinary differential equations, if one prescribes the initial value

$$u(x, 0) = h(x) \tag{2.4}$$

on the line $y = 0$, then in a convex Ω there will be a unique solution of the *initial value problem* (2.2) (2.4), namely, the solution is $u(x, y) = h(x)$ for all $(x, y) \in \mathbb{R}^2$. Again, one must be more careful for more complicated regions.

Exercise: Solve $u_y + u = 0$ with initial condition $u(x, 0) = 2x - 3$.

Instead of specifying the initial values on the line $y = 0$, one can prescribe them on a more general curve $\alpha(t) = (x(t),\, y(t))$, say

$$u(x(t),\, y(t)) = h(t). \tag{2.5}$$

In this case, using (2.3) one finds that

$$\varphi(x(t)) = h(t). \tag{2.6}$$

However, one cannot use an arbitrary curve α. For an extreme example, if α is vertical, that is, $x(t) =$ const., then one cannot solve the initial value problem (2.2) (2.5) unless $h(t) \equiv$const. Thus one cannot prescribe arbitrary initial data on an arbitrary curve. Even more seriously, if one differentiates (2.6), then one finds

$$\varphi'(x(t))\frac{dx}{dt} = \frac{dh}{dt}, \tag{2.7}$$

so if α is vertical for some value t_0, then $h'(t_0) = 0$.

MORAL: if α is tangent to a characteristic curve at some point, then one cannot solve (2.2) with initial condition (2.5) unless h satisfies some additional conditions. However, if α is nowhere nowhere tangent to a characteristic, then one can solve the problem — at least locally — given any h.

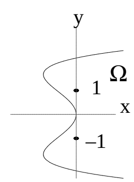
b). The inhomogeneous equation. One can readily extend this discussion to the inhomogeneous equation $u_y = f(x,y)$. The only new issue is finding one particular solution of the inhomogeneous equation $v_y = f$. Using this the initial value problem (2.1), (2.5) is reduced to the homogeneous case (2.2) (2.5) by letting $w = u - v$. Then w satisfies the homogeneous equation $w_y = 0$.

Exercise: Solve $u_y = 1 - 2xy$ with $u(x,0) = 0$.

If one attempts to find a particular solution of the inhomogeneous equation $u_y = f$ in a domain Ω, where $f \in \mathbb{C}^\infty(\Omega)$, then vertical convexity is again needed. In fact

PROPOSITION 2.1. *One can solve $u_y = f$ for all $f \in \mathbb{C}^\infty(\Omega)$ $\iff$ Ω is vertically convex.*

PROOF: $\Leftarrow$ Just integrate.
$\Rightarrow$ A proof can be found in [Hörmander-1, Theorems 3.5.4 and 3.7.2]. However the following argument (I learned it from G. Schwarz) is adequate for many domains — such as the region Ω in the figure. Let $f(x,y) = 1/r$, where $r = \sqrt{x^2+y^2}$. Assume there is a solution u of $u_y = f$. Then for any $\epsilon > 0$,



$$u(\epsilon,1) - u(\epsilon,-1) = \int_{-1}^{1} u_y(\epsilon,\,t)\,dt = \int_{-1}^{1} \frac{1}{\sqrt{\epsilon^2+t^2}}\,dt. \tag{2.8}$$

Now as $\epsilon \to 0$, the left side is finite but the right side becomes infinite. This contradiction completes the proof.

3. A More General Example

a). Constant coefficient. Other first order equations can be treated similarly. For example, the equation

$$u_x + 2u_y = 0, \tag{2.9}$$

can be interpreted as the directional derivative in the direction of the constant vector field $V(x,y) = (1,2)$ is zero: $V\cdot\nabla u = 0$. Thus u is constant along lines of slope 2, that is, on lines of the form $2x - y =$const. Thus $u(x,y)$ depends only on which straight line one is on, that is, on the value of $2x-y$. Hence $u(x,y) = h(2x-y)$ for some function $h(s)$. If we also ask, for instance, that u also satisfy the initial condition $u(x,0) = \sin x$, then $h(2x) = u(x,0) = \sin x$ so the solution is $u(x,y) = \sin(x - \frac{1}{2}y)$
The lines $2x - y =$const. are the *characteristics* of (2.9). There is an obvious analog of the vertically convex domains Ω for this equation.

ALTERNATE METHOD Another approach to (2.9) is to change to new coordinates so that (2.9) is the simpler $u_r = 0$. A linear change of variables is clearly appropriate.

$$r = ax + by$$
$$s = cx + dy$$

By the chain rule

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial x} = a\frac{\partial u}{\partial r} + c\frac{\partial u}{\partial s}$$
$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial y} = b\frac{\partial u}{\partial r} + d\frac{\partial u}{\partial s}$$

Thus

$$0 = u_x + 2u_y = (a+2b)u_r + (c+2d)u_s.$$

Since we want the form $u_r = 0$, let $c + 2d = 0$ and $a + 2b = 1$. Then $s = d(-2x+y)$. The choice of d is unimportant, so we just pick $d = 1$. The solution of $u_r = 0$ is $u(r,s) = \varphi(s) = \varphi(-2x+y)$ for any function φ. Using the initial condition we find

$$\sin x = u(x,0) = \varphi(-2x) \quad \text{so} \quad \varphi(x) = -\sin(x/2).$$

Consequently, just as above,

$$u(x,y) = \sin(x - \tfrac{1}{2}y).$$

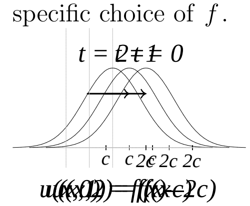
The *transport equation* is

$$u_t + cu_x = 0. \tag{2.10}$$

It is a simple model for the following situation. Say one has water flowing at a constant velocity c in a horizontal cylindrical pipe along the x-axis. Initially , near $x = 0$ a colored dye is inserted in the water. Ignoring possible dispersion of the dye, it will simply flow along the pipe. The concentration $u(x,t)$ of the dye, then is reasonably described by the transport equation. If the initial concentration is $u(x,0) = f(x)$, then by our discussion in the previous paragraph, the solution is

$$u(x,t) = f(x - ct).$$

This solution $f(x-ct)$ represents a "density wave" traveling to the right with velocity c. To see this we sketch $u(x,t) = f(x-ct)$ for a specific choice of f.



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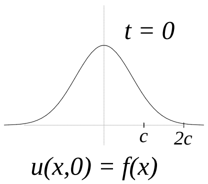


Exercise: Solve $u_x + u_y = 0$ with the initial value $u(0, y) = 3 \sin y$.

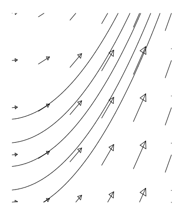
Exercise: Solve $u_x + u_y + 2u = 0$ with the initial value $u(0, y) = 3y$.

b). Variable coefficient. First order linear equations with variable coefficients

$$a(x, y)u_x + b(x, y)u_y = f(x, y) \tag{2.11}$$

are also easy to understand. Let $V(x, y)$ be the vector field

$$V(x, y) = (a(x, y), b(x, y)).$$

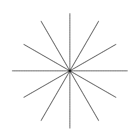


Then $au_x + bu_y$ is just the directional derivative of u along V. Let $(x(t), y(t))$ be the integral curves of this vector field

$$\frac{dx}{dt} = a(x, y), \qquad \frac{dy}{dt} = b(x, y). \tag{2.12}$$

In the homogeneous case $f = 0$, (2.11) means that $u(x, y)$ is a solution if and only if it is constant along these curves. These are the *characteristics* of (2.11). To solve (2.11) one introduces these characteristic curves as new coordinates. This will enable us to reduce the equation to the simple form (2.1).

Example 2.2. $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = f(x, y)$.



The integral curves of the vector field (x, y) are $x(t) = x_0 e^t$, $y(t) = y_0 e^t$. These are half-rays through the origin (the origin is a singular point of the vector field so for the present we delete it from consideration). These curves are the characteristics of our example. These radial lines tell us to introduce polar coordinates. Then the equation becomes simply

$$ru_r = f(r \cos\theta, r \sin\theta), \qquad \text{that is} \qquad u_r = \frac{f(r \cos\theta, r \sin\theta)}{r},$$

that is exactly of the form (2.1).

Exercise: Use this procedure to obtain solutions of $u_x - u_y = 0$. Find a solution satisfying the initial condition $u(x, 0) = xe^2x$. Is this solution unique?

Exercise: Solve $xu_x + u_y = 0$ with $u(x, 0) = g(x)$. Consider solving the same equation but with the initial condition $u(0, y) = h(y)$.

Exercise: Solve $yu_x - xu_y = xy$ with $u(x, 0) = 0$ for $x > 0$. Is your solution valid for all (x, y) other than the origin?

For two independent variables, the following small modification is sometimes convenient. Consider for the homogeneous equation

$$a(x, y)u_x + b(x, y)u_y = 0 \tag{2.13}$$

We find a solution using our previous observation that the solutions are precisely those functions that are constant along the characteristic curves. For convenience, say $a(x_0, y_0) \neq 0$ so near (x_0, y_0) (2.12) can be written as

$$\frac{dy}{dx} = \frac{b(x, y)}{a(x, y)}. \tag{2.14}$$

Write the solution of this that passes through the point $x = x_0$, $y = c$ as $y = g(x, c)$. Then of course $c = g(x_0, c)$. This curve is the characteristic that passes through (x_0, c). By the implicit function theorem the equation $y = g(x, c)$ can be solved for c to rewrite the equation of the characteristics in the form $\varphi(x, y) = c$. But $u = \varphi(x, y)$ is constant along these characteristic curves, as is $u = F(\varphi(x, y))$ for any function F. Thus $u(x, y) = F(\varphi(x, y))$ is a solution of the homogeneous equation (2.13) for any F.

Exercise: Use this procedure to find the general solution of $xu_x + bu_y = 1$. Here b is a constant. [Suggestion: To apply the method, first find (by inspection) a particular solution of the homogeneous equation.]

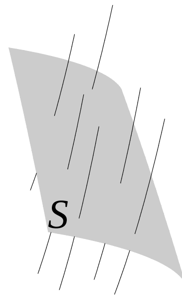
We next extend these ideas to n independent variables $x := (x_1, \ldots, x_n)$. Let $a_1(x), \ldots, a_n(x)$ be real functions. Then locally one can solve

$$Pu := \sum_{j=1}^{n} a_j(x) \frac{\partial u}{\partial x_j} = f(x) \tag{2.15}$$

by observing that Pu is the directional derivative of u in the direction of the vector field $V(x) := (a_1(x), \ldots, a_n(x))$, which is assumed nonsingular. Thus $Pu = f$ specifies the directional derivative of u along the integral curves to this vector field. These integral curves are the solutions of the ordinary differential equation

$$\frac{dx_j}{dt} = a_j(x(t)), \quad j = 1, \ldots, n; \qquad \text{that is} \qquad \frac{dx}{dt} = V(x). \tag{2.16}$$

If this vector field is differentiable, then (locally) through every point there is a a unique solution., this solution being the integral curve of V through the point. Let S be an $n-1$ dimensional surface that is transversal to these integral curves (*transversal* means that the integral curves are not tangent to S). Let $\xi := (\xi_1, \ldots, \xi_{n-1})$ be local coordinates on S. Pick the parameter t so that at $t = 0$ the integral curves are on S and let $x = x(t, \xi)$ be the integral curve passing through ξ when $t = 0$, so $x(0, \xi) = \xi \in S$. Introduce the new coordinates (t, ξ) in place of $(x_1, \ldots, x_n)$ and notice that $u_t = \sum_j \frac{\partial u}{\partial x_j}\frac{dx_j}{dt} = \sum_j a_j(x)\frac{\partial u}{\partial x_j}$. Thus (2.15) assumes the simple canonical form



$$\frac{\partial u}{\partial t} = f(t, \xi).$$

This is exactly the special case (2.1) we have already solved. After one has solved this, then one reverts to the original x coordinates.

Example: Solve $2u_x + yu_y - u_z = 0$ with $u(x, y, 0) = (x - y)^2$.

To solve this, write $V = (2, y, -1)$. Then the differential equation states that $V \cdot \nabla u$=0, that is, u is constant along the integral curves of this vector field v. These integral curves are the characteristics of the differential equation. To find the characteristics we integrate

$$\frac{dx}{dt} = 2, \quad \frac{dy}{dt} = y, \quad \frac{dz}{dt} = -1.$$

The solution is

$$x = 2t + \alpha, \quad y = \beta e^t \quad z = -t + \gamma,$$

where α, β, and γ are constants. Since the parameter t is arbitrary, we pick $t = 0$ when $z = 0$ This gives $\gamma = 0$, so $z = -t$. Then we can replace the parameter t by $-z$.

$$x = -2z + \alpha, \quad y = \beta e^{-z}.$$

The solution $u(x, y, z)$ depends only on the integral curve passing through the point (x, y, z), so it depends only on $\alpha = x + 2z$ and $\beta = ye^z$:

$$u(x, y, z) = h(x + 2z,\, ye^z)$$

for some function h which we now determine from the initial condition

$$(x - y)^2 = u(x, y, 0) = h(x, y).$$

Consequently,

$$u(x, y, z) = (x + 2z - ye^z)^2.$$

Exercise: Use this approach to solve $2u_x + u_y - xu_z = 2x$ with the initial condition $u(x, 0, z) = 0$.

Exercise: (Transport equation). Consider functions $u(x, t)$ in the $n+1$ variables $(x, t) := (x_1, \ldots, x_n, t)$ and let $c := (c_1, \ldots, c_n) \in \mathbb{R}^n$. Solve the transport equation in n space variable: $u_t + c_1 u_{x_1} + \cdots + c_n u_{x_n} = 0$, that is, $u_t + c \cdot \nabla u = 0$, with initial condition $u(x, 0) = F(x)$. You should be led to the solution $u(x, t) = F(x - ct)$.

Exercise: Discuss how to solve $a(x, y)u_x + b(x, y)u_y + c(x, y)u = f(x, y)$ by introducing the characteristics as coordinates and reducing to an equation of the form $u_t + p(t, \xi)u = h(t, \xi)$, which can be solved locally by ODE techniques.
Solve $2u_x + u_y - xu_z + u = 2x$ with the initial condition $u(x, 0, z) = 0$.

4. A Global Problem

a). Statement. So far we have limited our discussion to local and "semi-local" problems. Let $\mathbb{T}^2 = \{\, (x, y) \in \mathbb{R}^2 \colon 0 \le x \le 2\pi,\ 0 \le y \le 2\pi \,\}$ be the torus, where we identify $x = 0$ with $x = 2\pi$, and $y = 0$ with $y = 2\pi$. Then $C^\infty(\mathbb{T}^2)$ is just the set of smooth functions that are 2π periodic in both x and y. Let $\gamma \neq 0$ and c be a real constants. The problem is: given $f \in C^\infty(\mathbb{T}^2)$, find $u \in C^\infty(\mathbb{T}^2)$ so that

$$Lu := u_x - \gamma u_y + cu = f(x, y). \tag{2.17}$$

Without loss of generality we may assume that $\gamma > 0$, since if it is not, we can replace y by $-y$. For c there are two cases, $c > 0$ and $c = 0$ (if $c < 0$, replace x by $-x$ and y by $-y$).

If needed, refer to the Section 5 of this chapter for a speedy tour of Fourier series. They are essential here.

b). Application of Fourier Series to $u_x - \gamma u_y + cu = f(x, y)$.
Case 1: $c > 0$ We seek a solution u of (2.17) as a Fourier series

$$u(x, y) = \sum_{k,\ell} u_{k\ell} e^{i(kx + \ell y)}.$$

Differentiating term-by-term we obtain

$$Lu = \sum i(k - \gamma\ell) u_{k\ell} e^{i(kx + \ell y)}.$$

Thus, if u is to satisfy (2.17) $Lu = f$, matching the above Fourier series for Lu with that (2.33) for f and using the orthogonality of $e^{i(kx+\ell y)}$, we find the equation

$$i(k - \gamma\ell + c)u_{k\ell} = f_{k\ell}, \quad k, \ell = 0, \pm 1, \pm 2, \ldots \tag{2.18}$$

Thus,

$$u_{k\ell} = \frac{f_{k\ell}}{i(k - \gamma\ell) + c}, \qquad \text{so} \qquad u(x, y) = \sum \frac{f_{k\ell}}{i(k - \gamma\ell) + c} e^{i(kx + \ell y)}.$$

Since $c > 0$ then $|\, i(k-\gamma\ell)+c\,| \geq c > 0$ so $|\, u_{k\ell}\,| \leq |\, f_{k\ell}\,|/c$. Thus, by (2.39) if $f \in C^\infty(\mathbb{T}^2)$ then $u \in C^\infty(\mathbb{T}^2)$.

Case 2: $c = 0$* If we integrate both sides of the equation over $\mathbb{T}^2$, by the periodicity of u we immediately find the necessary condition

$$\int_{\mathbb{T}^2} f(x,y)\,dx\,dy = 0.$$

We seek a solution u of (2.17) as

$$u(x,y) = \sum_{k,\ell} u_{k\ell} e^{i(kx+\ell y)}.$$

Formally, after differentiating term-by-term we obtain

$$Lu = \sum i(k-\gamma\ell) u_{k\ell} e^{i(kx+\ell y)}.$$

Thus, if u is to satisfy (2.17) $Lu = f$, matching the above Fourier series for Lu with that (2.33) for f and using the orthogonality of $e^{i(kx+\ell y)}$, we find the equation

$$i(k-\gamma\ell)u_{k\ell} = f_{k\ell}, \quad k,\ell = 0,\pm1,\pm2,\ldots \tag{2.19}$$

If $k = \ell = 0$ this implies $f_{00} = 0$, which is just $\int_{\mathbb{T}^2} f = 0\,dx$ (again). Moreover, if $\gamma = p/q$ is rational, then $f_{k\ell} = 0$ whenever $k/\ell = p/q$. This gives infinitely many conditions on f. We will not pursue this case further and consider only the case when γ is *irrational*. Then solving (2.19) for $u_{k\ell}$ and using them in the Fourier series for u we obtain

$$u(x,y) = \sum \frac{-if_{k\ell}}{k-\gamma\ell}\, e^{i(kx+\ell y)}. \tag{2.20}$$

It remains to consider the convergence of this series. We'll use Lemma 2.39 to determine if u is smooth. It is clear that there will be trouble if γ can be too-well approximated by rational numbers, since then the denominator $\gamma-(k/\ell)$ will be small. This is the classical problem of *small divisors*. Of course every real number can be closely approximated by a rational number p/q. The issue is how large the denominator q must be to get a good approximation.

DEFINITION 2.3. An irrational number γ is a *Liouville number* if for every positive integer ν and any $k > 0$ then $\left|\dfrac{p}{q}-\gamma\right| < \dfrac{k}{q^\nu}$ for infinitely many pairs of integers (p,q). Thus, γ is *not* a Liouville number if there exist numbers ν and k so that $\left|\dfrac{p}{q}-\gamma\right| > \dfrac{k}{q^\nu}$ for all but a finite number of integers p, q.

*The results in the remainder of this section will not be used elsewhere in these notes.

REMARK 2.4. If γ is a real algebraic number of degree $m \geq 2$ over the rational numbers, then it is not a Liouville number and one may pick $\nu = n$. Here is the proof. Say the real irrational number γ is a root of

$$h(x) := a_n x^n + \cdots + a_0 = 0$$

with integer coefficients and $a_n \neq 0$ and that p/q is so close to γ that $h(p/q) \neq 0$. Then

$$\left|h\left(\frac{p}{q}\right)\right| = \frac{|\, a_n p^n + a_{n-1}p^{n-1}q + \cdots\,|}{q^n} \geq \frac{1}{q^n}$$

since the numerator is a non-zero integer. Thus by the mean value theorem

$$\frac{1}{q^n} \leq \left|h\left(\frac{p}{q}\right)\right| = \left|h\left(\frac{p}{q}\right) - h(\gamma)\right| = \left|\gamma - \frac{p}{q}\right| |\, h'(c)\,|$$

for some c between γ and p/q. Thus,

$$\left|\gamma - \frac{p}{q}\right| \geq \frac{M}{q^n}, \tag{2.21}$$

where $M = 1/|\, h'(c)\,|$.
Liouville used this approach to exhibit the first transcendental number around 1850; only later were e and π proved to be transcendental. The inequality (2.21) was subsequently improved successively by Thue, Siegel, Dyson, Gelfond, and Roth. Roth's final result is that the exponent n on the right side of (2.21) can be replaced by 2. He was awarded a Fields Medal for this.

Exercise: Show that $\alpha := \sum 2^{-n!}$ and $\beta := \sum 10^{-n!}$ are Liouville numbers and hence transcendental.

Exercise: Show that the set of Liouville numbers $0 < \gamma < 1$ has measure zero.

We are now in a position to prove the following striking result on the global solvability of (2.17) on the torus. It will be convenient to use the following equivalent definition of a Liouville number:

LEMMA 2.5. *γ is a Liouville number if for every positive integer ν and any $k > 0$ then $\left|\dfrac{p}{q}-\gamma\right| < \dfrac{k}{(1+p^2+q^2)^{\nu/2}}$ for infinitely many pairs of integers (p,q).*

Proof: Since $1/(1+p^2+q^2)^{\nu/2} < 1/q^\nu$, a Liouville number in this sense also satisfies the previous definition. Conversely, if γ is a Liouville number in the previous sense, then for any integer $\nu > 0$ and any $k > 0$, we know $|\,\gamma - p/q\,| < 1/q^\nu$ for infinitely many p, q. This, with $\nu = 0$ and $k = 1$, gives the crude estimate $|\,p\,| \leq |\,p-\gamma q\,| + |\,\gamma q\,| \leq 1 + \gamma q$

so $1+p^2+q^2 \le c(1+q^2)$ (we can let $c = 3+2\gamma^2$, but this value is unimportant). Since $q \ge 1$,

$$\frac{1}{q^2} \le \frac{2}{1+q^2} \le \frac{2c}{1+p^2+q^2}.$$

Thus, if γ is a Liouville number in the previous sense then it is also a Liouville by this alternate definition.

THEOREM 2.6. *Let γ be an irrational number. Then the equation* (2.17) *has a solution $u \in \mathbb{C}^\infty(\mathbb{T}^2)$ for all $f \in \mathbb{C}^\infty(\mathbb{T}^2)$ if and only if γ is not a Liouville number.*

PROOF: Say γ is irrational but not a Liouville number. Our goal is to estimate the growth of the Fourier coefficients in (2.20) and show the series converges to some smooth u. Since γ is not a Liouville number, there is an integer $\nu > 0$ and a number k such that for all but a finite number of integers p, q we have

$$\left|\gamma - \frac{p}{q}\right| > \frac{k}{(1+p^2+q^2)^{\nu/2}}.$$

This will allow us to estimate the denominators in (2.20). To estimate the numerators we use the above Lemma 2.5. Consequently, for any s there is come constant $c(s)$ so that

$$|\, u_{k\ell}\,| = \frac{|\, f_{k\ell}\,|}{|\, k-\gamma\ell\,|} \le \frac{c(s)(1+k^2+\ell^2)^{\nu/2}}{(1+k^2+\ell^2)^s} = \frac{c(s)}{(1+k^2+\ell^2)^{s-(\nu/2)}}.$$

Since s is arbitrary, by the Lemma again we find that u is smooth. Therefore we can differentiate (2.20) term-by-term and verify that it satisfies the differential equation (2.17).

Conversely, if γ is a Liouville number, we will exhibit a smooth f so that with this f the equation (2.17) has no smooth solution. Since γ is Liouville, for any $k > 0$, $\nu > 0$ there are infinitely many pairs of integers (p,q) that satisfy

$$|\, p-\gamma q\,| < \frac{k}{(1+p^2+q^2)^{\nu/2}}$$

Using this with $\nu = 2j$ and $k = 1$, for each j pick one point (p_j, q_j).

$$|\, p_j - \gamma q_j\,| < \frac{1}{(1+p_j^2+q_j^2)^j} \tag{2.22}$$

We may assume that $p_j^2+q_j^2 < p_{j+1}^2+q_{j+1}^2$ to insure that each of these lattice points selected is associated with only one index j. Define f by setting

$$f_{p_jq_j} = \frac{1}{(1+p_j^2+q_j^2)^j}$$

for these lattice points (p_j, q_j) while for all other lattice points (k,ℓ) we set $f_{k\ell} = 0$. Then by Lemma 2.5 f is smooth. However from (2.19) and (2.22)

$$|\, u_{p_jq_j}\,| = \frac{|\, f_{p_jq_j}\,|}{|\, p_j-\gamma q_j\,|} > 1$$

so u is not smooth. In fact, u is not even in $L^2(\mathbb{T}^2)$. Consequently if γ is a Liouville number, then there is no smooth solution.

5. Appendix: Fourier series

Many problems in science and technology lead naturally lead one to Fourier series. They are a critical tool in these notes.

a). Fourier series on S^1. Say a function $f(x)$ is periodic with period 2π. It is useful to think of these as functions on the unit circle, S^1. The simplest functions with this periodicity are e^{ikx}, $k = 0, \pm1, \pm2 \ldots$ (or, equivalently, $\cos kx$ and $\sin kx$). One tries to write f as a linear combination of these functions

$$f(x) \sim \sum_{\ell=-\infty}^{\infty} a_\ell e^{i\ell x}. \tag{2.23}$$

But how can you find the coefficients a_ℓ? What saves the day (and was implicitly realized by Euler as well as Fourier) is to introduce the *inner product*

$$\langle \varphi,\, \psi\rangle = \int_{-\pi}^{\pi} \varphi(x)\overline{\psi(x)}\, dx,$$

and say that φ is *orthogonal* to ψ when $\langle \varphi,\, \psi\rangle = 0$. Note that if φ and ψ are orthogonal, then the Pythagorean formula is valid:

$$\|\varphi+\psi\|^2 = \|\varphi\|^2 + \|\psi\|^2.$$

In this inner product e^{ikx} and $e^{i\ell x}$ are orthogonal for integers $k \ne \ell$. As in $\mathbb{R}^n$ we also write the norm

$$\|\varphi\| = \langle \varphi,\, \varphi\rangle^{1/2} = \left[\int_{-\pi}^{\pi} |\, \varphi(x)\,|^2\, dx\right]^{1/2} \tag{2.24}$$

[to keep history in perspective, the inner product in $\mathbb{R}^n$ was introduced only in the late nineteenth century]. Since $\|e^{ikx}\|^2 = \langle e^{ikx},\, e^{ikx}\rangle = 2\pi$, it is convenient to use the *orthonormal* functions $e^{ikx}/\sqrt{2\pi}$ and write

$$f(s) \sim \sum_{\ell=-\infty}^{\infty} c_\ell \frac{e^{i\ell x}}{\sqrt{2\pi}}. \tag{2.25}$$

Formally taking the inner product of both sides of this with $e^{ikx}/\sqrt{2\pi}$ we obtain the classical formula for the Fourier coefficients

$$c_k = \langle f,\, \frac{e^{ikx}}{\sqrt{2\pi}}\rangle = \int_{-\pi}^{\pi} f(x)\frac{e^{-ikx}}{\sqrt{2\pi}}\, dx. \tag{2.26}$$

Understanding the convergence of the Fourier series (2.25) is fundamental. This convergence clearly depends on the decay of the Fourier coefficients c_ℓ. First we discuss convergence in the norm (2.24).

Let $\mathcal{T}_N$ be the (finite dimensional) space of trigonometric polynomials whose degree is at most N, that is, these functions have the form $t_N(x) = \sum_{|k|\le N} a_k e^{ikx}$. Also let $P_N(f) := \sum_{|k|\le N} c_k \frac{e^{ikx}}{\sqrt{2\pi}} \in \mathcal{T}_N$ be the terms in (2.25) with $|k| \le N$. By (2.26), note that $f - P_N(f)$ is orthogonal to $\mathcal{T}_N$ because if $|\ell| \le N$ then $\langle f - P_N(f), e^{i\ell x}\rangle = 0$. Thus, we have written

$$f = P_N(f) + [f - P_N(f)]$$

as the sum of a function in $\mathcal{T}_N$ and a function orthogonal to $\mathcal{T}_N$ so we call $P_N(f)$ the *orthogonal projection* of f in the subspace $\mathcal{T}_N$. Since both $P_N(f)$ and t_N are in $\mathcal{T}_N$, then by the Pythagorean theorem

$$\|f - t_N\|^2 = \|f - P_N(f)\|^2 + \|P_N(f) - t_N\|^2 \ge \|f - P_N(f)\|^2. \tag{2.27}$$

In other words, in this norm *the function $P_N(f)$ is closer to f than any other function t_N in $\mathcal{T}_N$.*

The useful *Bessel's inequality* is a special case of the computation (2.27) when we pick $t_N \equiv 0$. It says

$$\|f\|^2 \ge \|P_N(f)\|^2 = \sum_{|k|\le N} |c_k|^2. \tag{2.28}$$

In particular, if f is piecewise continuous, so $\|f\| < \infty$, then $\sum_k |c_k|^2$ converges.

We will use the observation (2.27) to prove that if $f \in C(S^1)$, that is, if f is continuous and 2π periodic, then $P_N(f)$ converges to f in our norm (2.24). For this we use the *Weierstrass approximation theorem* to uniformly approximate f by some trigonometric polynomial $t_N(x) = \sum_{|k|\le N} a_k e^{ikx}$. Thus, given $\varepsilon > 0$ there is some trigonometric polynomial $t_N(x)$ so that $\max_{x\in S^1}|f(x) - t_N(x)| < \varepsilon$ (here N is determined by ε). This implies that $\|f - t_n\| \le \sqrt{2\pi}\,\varepsilon$. Consequently (2.27) gives the desired convergence in this norm:

$$\|f - P_N(f)\| \le \|f - t_N\| \le \sqrt{2\pi}\,\varepsilon, \tag{2.29}$$

that is, $\lim_{N\to\infty}\|f - P_N(f)\| \to 0$. Since $\|f\|^2 = \|P_N(f)\|^2 + \|f - P_N(f)\|^2$, this also implies the *Parseval identity*

$$\|f\|^2 = \lim_{N\to\infty}\|P_N(f)\|^2, \quad \text{that is} \quad \int_{-\pi}^{\pi}|f|^2 = \sum_k |c_k|^2. \tag{2.30}$$

Remark 2.7. While this reasoning used that $f \in C(S^1)$, it is straightforward to see that the results hold only assuming f is piecewise continuous (or even square integrable). For this we use that in the norm (2.24) one can approximate a piecewise continuous function on $[-\pi,\pi]$ by a continuous 2π periodic function.

Exercise: Let $f(x) := x$ for $-\pi \le x \le \pi$. Compute its Fourier series and the consequent formula that Parseval's identity (2.30) gives.

To obtain the uniform convergence of (2.25) we will prove that if f is smooth enough then the series $\sum|c_k|$ converges. By the Weierstrass M-Test this will give the uniform convergence. Thus we need to discuss the decay of the Fourier coefficients c_ℓ.

To understand this decay, without worrying about convergence formally take the derivative of both sides of (2.25) to find that

$$f'(s) \sim \sum_{\ell=-\infty}^{\infty} i\ell c_\ell \frac{e^{i\ell x}}{\sqrt{2\pi}}. \tag{2.31}$$

Thus, we suspect that the Fourier coefficients of f' are ikc_k. This is easy to prove directly if f is periodic and has a continuous derivative; just use integration by parts in (2.26) to obtain

$$c_k = \int_{-\pi}^{\pi} f'(x)\frac{e^{-ikx}}{ik\sqrt{2\pi}}\,dx.$$

Consequently,

$$|c_k| \le \frac{\sqrt{2\pi}}{|k|}\max|f'(x)|.$$

Repeating this procedure, we find that if $f \in C^j(S^1)$ and, with its derivatives, is periodic, then

$$|c_k| \le \frac{\sqrt{2\pi}}{|k|^j}\max|D^j f(x)|.$$

Thus, the smoother f is, the faster its Fourier coefficients decay. In particular, if $f \in C^2(S^1)$ (so f, f', and f'' are periodic), then $|c_k| \le \text{const}/k^2$ so the series $|c_k|$ converges and hence the Fourier series (2.25) converges uniformly to f.

By being more careful, we can prove that the Fourier series converges uniformly if $f \in C^1(S^1)$; in fact, all we will really require is that f' is square integrable. For this we use Bessel's inequality (2.28) applied to f':

$$\sum_k |kc_k|^2 \le \|f'\|^2. \tag{2.32}$$

Therefore, by the Schwarz inequality,

$$\sum_{|k|\le N} |c_k| = \sum_{|k|\le N} \frac{1}{\sqrt{1+|k|^2}}\sqrt{1+|k|^2}\,|c_k|$$
$$\le \left[\sum_{|k|\le N}\frac{1}{1+|k|^2}\right]^{1/2}\left[\sum_{|k|\le N}(1+|k|^2)|c_k|^2\right]^{1/2}.$$

The second series converges by (2.32) (in fact, it converges to $\left[\|f\|^2 + \|f'\|^2\right]^{1/2}$), while the first by comparison to $\sum 1/|k|^2$.

Exercises:

1. Let c_k be the Fourier coefficients of $f \in C(S^1)$. Show that if f and all of its derivatives exist and are continuous, then for any integer $s \ge 0$ there is a constant $M(s)$ so that $|c_k| \le M(s)/(1+|k|^2)^{s/2}$.

2. Conversely, if for any integer $s \ge 0$ there is a constant $M(s)$ so that $|c_k| \le M(s)/(1+|k|^2)^{s/2}$, show that $f \in C^\infty(S^1)$.

3. If $f \in C^1(S^1$, use Fourier series to solve $-u'' + u = f$ on S^1.

4. If $f \in C^1(S^1$, use Fourier series to discuss when one can solve $-u'' = f$ on S^1.

b). Fourier series on $\mathbb{T}^n$. With the above theory for Fourier series in one variable as motivation, we now investigate Fourier series in n variables, that is, on the n-dimensional torus $\mathbb{T}^n$. As a warm-up, in two variables we write

$$f(x,y) = \sum_{k,\ell} f_{k\ell}\, e^{i(kx+\ell y)} \tag{2.33}$$

where the Fourier coefficients are given by

$$f_{k\ell} = \tfrac{1}{(2\pi)^2}\int_{\mathbb{T}^2} f(x,y)e^{-i(kx+\ell y)}\,dx\,dy.$$

Note that we have switched normalization from (2.25) to (2.23).

For n variables, to avoid a mess we introduce some notation. Write $x = (x_1,\dots,x_n) \in \mathbb{T}^n$, let $k = (k_1,\dots,k_n)$ be a multi-index (vector) with integer coefficients, $|k| = [k_1^2 + \cdots, k_n^2]^{1/2}$ and let $k\cdot x = k_1x_1 + \cdots, k_nx_n$. Then the (formal) Fourier series for f is

$$f(x) = \sum_k f_k\, e^{ik\cdot x} \quad \text{where} \quad f_k = \frac{1}{(2\pi)^n}\int_{\mathbb{T}^n} f(x)e^{-ik\cdot x}\,dx. \tag{2.34}$$

Parseval's theorem states that if f is square integrable, then in the L^2 norm $\|f\|^2 := \int_{\mathbb{T}^n}|f(x)|^2\,dx$ we have

$$\|f\|^2 = (2\pi)^n\sum |f_k|^2.$$

We next generalize the insight we found with Fourier series in one variable that the smoothness of a function is encoded in the decay of its Fourier coefficients. If u has a Fourier series

$$u(x) = \sum_k u_k\, e^{ik\cdot x} \tag{2.35}$$

then, formally,

$$(-\Delta + 1)u(x) = \sum_k (1+|k|^2)u_k\, e^{ik\cdot x}. \tag{2.36}$$

Using this and the divergence theorem we observe that for a real function u

$$\int_{\mathbb{T}^n}\left[|u|^2 + |\nabla u|^2\right]dx = \int_{\mathbb{T}^n}\left[u^2 - u\Delta u\right]dx = \sum_k (1+|k|^2)|u_k|^2.$$

and

$$\int_{\mathbb{T}^n}\left[|u|^2 + 2|\nabla u|^2 + |\Delta u|^2\right]dx = \int_{\mathbb{T}^n}\left[u(1-\Delta)^2u\right]dx = \sum_k (1+|k|^2)^2|u_k|^2$$

(for complex functions u one just adds a few complex conjugate signs). Using this as motivation, define the *Sobolev spaces* $H^s(\mathbb{T}^n)$ to be the space of functions φ with finite norm

$$\|\varphi\|^2_{H^1(\mathbb{T}^n)} := \sum_k (1+|k|^2)^s|\varphi_k|^2 < \infty.$$

Of course $H^0(\mathbb{T}^n) = L^2(\mathbb{T}^n)$.

One thinks of $H^s(\mathbb{T}^n)$ as the space of functions on $\mathbb{T}^n$ whose derivatives up to order s are square integrable. To see this, let $r = (r_1,\dots,r_n)$ be any multi-index of integers with $\sum r_j = r$ and let $D^r = (\partial/\partial x_1)^{r_1}\cdots(\partial/\partial x_n)^{r_n}$ be a partial derivative of order $r \le s$. Then, using

$$\varphi(x) = \sum_k \varphi_k e^{ik\cdot x} \tag{2.37}$$

we have

$$D^r\varphi(x) = \sum_k (i)^r(k_1^{r_1}\cdots k_n^{r_n})\varphi_k e^{ik\cdot x}.$$

But $|k_1^{r_1}\cdots k_n^{r_n}| \le |k|^r$ so

$$\|D^r\varphi\|^2_{L^2} \le \sum_k |k|^{2r}|\varphi_k|^2.$$

Because $r \le s$ we have $|k|^{2r} \le (1+|k|^2)^s$ so the above sum is finite.

It should be clear that if $f \in C^s(\mathbb{T}^n)$, that is, if the derivatives of f up to order s are continuous and periodic, then $f \in H^s(\mathbb{T}^n)$ but $H^s(\mathbb{T}^n)$ is a much larger space. However, we will show that if $\varphi \in H^s(T^n)$ for sufficiently large s, then φ is continuous. Using the Weierstrass

M-test, from (2.37) it is enough to show that $\sum|\varphi_k| < \infty$. But by the Schwarz inequality

$$\begin{aligned}\sum|\varphi_k| &= \sum_k \frac{1}{(1+|k|^2)^{s/2}}(1+|k|^2)^{s/2}|\varphi_k| \\ &\le \left[\sum_k \frac{1}{(1+|k|^2)^s}\right]^{1/2}\left[\sum_k (1+|k|^2)^s|\varphi_k|^2\right]^{1/2} \\ &= \left[\sum_k \frac{1}{(1+|k|^2)^s}\right]^{1/2}\|\varphi\|_{H^2(\mathbb{T}^n)}.\end{aligned}$$

The series $\sum 1/(1+|k|^2)^s$ converges for all $s > n/2$. One way to see this is by comparison with an integral using polar coordinates

$$\int_{\mathbb{R}^n} \frac{dx}{(1+|x|^2)^s} = \text{Area}\,(S^{n-1})\int_0^\infty \frac{r^{n-1}\,dr}{(1+r^2)^s}.$$

This integral converges if $2s-(n-1) > 1$, that is, if $s > n/2$. Thus, if $s > n/2$, there is a constant c so that if $\varphi \in H^s(\mathbb{T}^n)$ then

$$\|\varphi\|_{C^0(\mathbb{T}^n)} \le c\|\varphi\|_{H^s(\mathbb{T}^n)}.$$

One consequence is that if we have a Cauchy sequence in $H^s(\mathbb{T}^n)$ and if $s > n/2$, then it is Cauchy in $C^0(\mathbb{T}^n)$ and hence converges uniformly to a continuous function. This is expressed as the *Sobolev embedding theorem*: if $s > n/2$, then $C^0 \subset H^s$.

If we apply this to a j^{th} derivative of φ, we find the following basic result.

Theorem 2.8. Sobolev inequality

$$\text{(2.38)} \qquad \textit{If} \quad s > j + n/2 \quad \textit{then} \quad \|\varphi\|_{C^j(\mathbb{T}^n)} \le c\|\varphi\|_{H^s(\mathbb{T}^n)}$$

and corresponding embedding theorem:

Theorem 2.9. Sobolev embedding theorem *If $s > j + n/2$ then $C^j \subset H^s$.*

This shows that

Corollary 2.10. *If φ is in $H^s(\mathbb{T}^2)$ for all positive integers s, then φ is smooth: $\varphi \in C^\infty(\mathbb{T}^n)$. That is, $C^\infty(\mathbb{T}^n) = \cap_s H^s(\mathbb{T}^n)$. Equivalently, $\varphi \in \mathbb{C}^\infty(\mathbb{T}^n)$ if and only if its Fourier coefficients decay faster than any polynomial: for any integer $s \ge 0$ there is a constant $c(s)$ so that*

$$\text{(2.39)} \qquad |\varphi_k| \le \frac{c(s)}{(1+|k|^2)^{s/2}}.$$

CHAPTER 3

The Wave Equation

1. Introduction

Light and sound are but two of the phenomena for which the classical wave equation is a reasonable model. This study is one of the real success stories in mathematics and physics. It has led to the development of many valuable techniques.

2. One space dimension

Upon studying the motion of a vibrating string one is led to the simple differential equation

$$u_{tt} = c^2 u_{xx}, \tag{3.1}$$

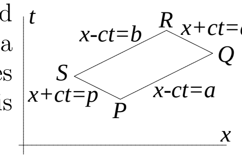
where $u(x,t)$ denotes the displacement of the string at the point x at time t and $c > 0$ is a constant that involves the density and tension of the string. We'll shortly show how to interpret c as the velocity of the propagation of the wave.
By making the change of variables $\xi = x - ct$ and $\eta = x + ct$ in (3.1), we find

$$u_{\xi\eta} = 0.$$

Integrating this twice reveals the "general" solution $u(\xi,\eta) = f(\xi) + g(\eta)$ for any twice differentiable functions f and g. Untangling the change of variables give us the general solution of (3.1):

$$u(x,t) = F(x - ct) + G(x + ct). \tag{3.2}$$

The term $F(x - ct)$ represents a wave traveling to the right with velocity c. We saw this in the previous Section a) when we discussed the transport equation. The sketches there substantiate the statement that c is the velocity of propagation of the wave. Similarly, $G(x+ct)$ represents a wave traveling to the left with velocity c, so the general solution is composed of waves traveling in both directions. The two families of straight lines $x - ct =$const, and $x + ct =$const are the *characteristics* of the wave equation (3.1).



The formula (3.2) implies an interesting identity we will need soon. Let P, Q, R, and S be the successive vertices of a parallelogram whose sides consist of the four characteristic lines $x - ct = a$, $x - ct = b$, $x + ct = p$, and $x + ct = q$. If $u(x,t)$ is a solution of the wave equation, then

$$u(P) + u(R) = u(Q) + u(S). \tag{3.3}$$

This is clear since $u(P) = F(a) + G(p)$, $u(Q) = F(a) + G(q)$, $u(R) = F(b) + G(q)$, and $u(S) = F(b) + G(p)$.

a). Infinite string, $-\infty < x < \infty$. On physical grounds based on experiments with the motion of particles, we anticipate that we should specify the following *initial conditions*:

$$\begin{array}{ll} \text{initial position} & u(x,0) = f(x) \\ \text{initial velocity} & u_t(x,0) = g(x). \end{array} \tag{3.4}$$

Using these conditions we can uniquely determine F and G in (3.2). This gives *d'Alembert's solution* of the initial value problem (3.1), (3.4):

$$u(x,t) = \frac{f(x+ct) + f(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds. \tag{3.5}$$

Exercise: Consider the equation

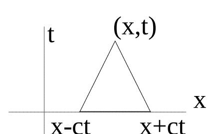
$$u_{xx} - 3u_{xt} - 4u_{tt} = 0. \tag{3.6}$$

a) Find a change of variable $\xi = ax + bt$, $\eta = cx + dt$ so that in the new coordinates the equation is the standard wave equation

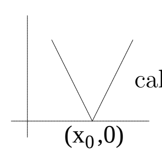
$$u_{\xi\xi} = u_{\eta\eta}.$$

b) Use this to solve (3.6) with the initial conditions

$$u(x,0) = x^2,\ u_t(x,0) = 2e^x.$$



It is instructive to note that the solution at (x,t) depends only on the initial data in the interval between the points $x - ct$ and $x + ct$. This interval is called the *domain of dependence* of the point (x,t).



Similarly, the initial data at a point $(x_0, 0)$ can only affect the solution $u(x,t)$ for points in the triangular region $|\,x - x_0\,| \le ct$. This region is called the *domain of influence* of the point $(x_0, 0)$

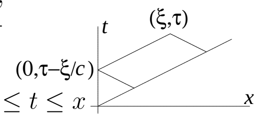
b). Semi-infinite string, $0 < x < \infty$**.** Semi-infinite strings can also be treated.

Special Case 1. As an example, we specify zero initial position and velocity but allow motion of the left end point:

$$u(x,0) = 0, \quad u_t(x,0) = 0 \text{ for } x > 0, \qquad \text{while } u(0,t) = h(t) \text{ for } t > 0. \tag{3.7}$$

We'll assume that $h(0) = 0$ to insure continuity at the origin.

The critical characteristic $x = ct$ is important here. The domain of dependence of any point to the right of this line does not include the positive t-axis. Thus, if $x \geq ct$, then $u(x,t) = 0$. Next we consider a point (ξ, τ) above this characteristic. The simplest approach is to use the identity (3.3) with a characteristic parallelogram having its base on the critical characteristic $x = ct$. The characteristic of the form $x - ct =$ const. through (ξ, τ) intersects the t-axis at $t = \tau - \xi/c$. Since $u(x,t) = 0$ on the base of this parallelogram, then by (3.3) we conclude that $u(\xi, \tau) = h(\tau - \xi/c)$. To summarize, we see that



$$u(x,t) = \begin{cases} 0 & \text{for} \quad 0 \leq t \leq x \\ h(t - x/c) & \text{for} \quad 0 \leq x \leq t. \end{cases}$$

Special Case 2. A clever observation helps to solve the related problem for a semi-infinite string:

$$u(x,0) = f(x), \quad u_t(x,0) = g(x) \text{ for } x > 0, \qquad \text{while } u(0,t) = 0 \text{ for } t > 0. \tag{3.8}$$

The observation is that for the infinite string $-\infty < x < \infty$, if the initial position $u(x,0) = f(x)$ and velocity $u_t(x,0) = g(x)$ are odd functions, then so is the solution $u(x,t)$ (proof?). Thus, to solve (3.8) we simply extend $f(x)$ and $g(x)$ to all of $\mathbb{R}$ as odd functions $f_{odd}(x)$ and $g_{odd}(x)$ and then use the d'Alembert formula (3.5).

Exercise: Carry this out explicitly for the special case where (3.8) holds with $g(x) = 0$. In particular, show that for $x > 0$ and $t > 0$

$$u(x,t) = \begin{cases} \frac{1}{2}[f(x+ct) + f(x-ct)] & \text{for} \quad x > ct \\ \frac{1}{2}[f(ct+x) - f(ct-x)] & \text{for} \quad x < ct. \end{cases}$$

The boundary condition at $x = 0$ serves as a reflection. One can see this clearly from a sketch, say with the specific function $f(x) = (x-2)(3-x))$ for $2 \leq x \leq 3$ and $f(x) = 0$ for both $0 \leq x \leq 2$ and $x > 3$.

General Case. For a semi-infinite string, the general problem with the initial and boundary conditions

$$u(x,0) = f(x), \quad u_t(x,0) = g(x), \text{ for } x > 0, \qquad \text{while } u(0,t) = h(t) \text{ for } t > 0$$

can now be solved by simply adding the solutions from the two special cases (3.7) (3.8) just treated.

Exercise: For the semi-infinite string $0 < x$, solve the initial-boundary value problem where the end at $x = 0$ is free (Neumann boundary condition):

$$u(x,0) = f(x), \quad u_t(x,0) = g(x) \text{ for } x > 0, \qquad \text{while } u_x(0,t) = 0 \text{ for } t > 0.$$

c). Finite string: $0 < x < L$**.** In the case of a finite string, such as a violin string, one must evidently also say something about the motion of the end points $x = 0$ and $x = L$. One typical situation is where we specify the position of these boundary points:

$$\text{left end: } \; u(0,t) = \varphi(t), \qquad \text{right end: } \; u(L,t) = \psi(t). \tag{3.9}$$

Thus, if the ends are tied down we would let $f(t) = g(t) = 0$. The equations (3.9) are called *boundary conditions*. As an alternate, one can impose other similar boundary conditions. Thus, if the right end is allowed to move freely and the left end is fixed ($\varphi(t) = \psi(t) = 0$), then the above boundary conditions become

$$u(0,t) = 0 \qquad \frac{\partial u}{\partial x}(L,t) = 0, \tag{3.10}$$

The condition at $x = L$ asserts the slope is zero there (that the slope at a free end is zero follows from physical considerations not given here). There is no simple "closed form" solution of the mixed initial-boundary value problem (3.1),(3.4), (3.9), even in the case $f(t) = g(t) = 0$. The standard procedure one uses is *separation of variables* (see section c) below). The solution is found as a Fourier series.

d). Conservation of Energy. For both physical and mathematical reasons, it is important to consider the energy in a vibrating string. Here we work with an infinite string.

$$E(t) = \tfrac{1}{2} \int_{-\infty}^{\infty} (u_t^2 + c^2 u_x^2)\, dx \tag{3.11}$$

The term u_t^2 is for the kinetic energy and $c^2 u_x^2$ the potential energy. (Here we have assumed the mass density is 1; otherwise $E(t)$ should be multiplied by that constant.) For this integral to converge, we need to assume that u_t and u_x decay fast enough at $\pm\infty$. From the d'Alembert formula (3.5), this follows if the initial conditions decay at infinity.
We prove *energy is conserved* by showing that $dE/dt = 0$. This is a straightforward computation involving one integration by parts — in

which the boundary terms don't appear because of the decay of the solution at infinity.

$$\frac{dE}{dt} = \int_{-\infty}^{\infty} (u_t u_{tt} + c^2 u_x u_{xt})\, dx = \int_{-\infty}^{\infty} u_t(u_{tt} - c^2 u_{xx})\, dx = 0, \tag{3.12}$$

where in the last step we used the fact that u is a solution of the wave equation.

Exercises

1. For a finite string $0 < x < L$ with zero boundary conditions: $u(0,t) = u(L,t) = 0$, define the energy as

$$E(t) = \tfrac{1}{2}\int_0^L (u_t^2 + c^2 u_x^2)\, dx. \tag{3.13}$$

Show that energy is conserved. Show that energy is also conserved if one uses the free boundary condition $\partial u/\partial x = 0$ at either — or both — endpoints.

2. For a finite string $0 < x < L$ let u be a solution of the modified wave equation

$$u_{tt} + b(x,t)u_t = u_{xx} + a(x,t)u_x \tag{3.14}$$

with zero Dirichlet boundary conditions: $u(0,t) = u(L,t) = 0$, where we assume that $|\,a(x,t)\,|,\ |\,b(x,t)\,| < M$ for some constant M. Define the energy by (3.13).
   a) Show that $E(t) \le e^{\alpha t}E(0)$ for some constant α depending only on M. [Suggestion: Use the inequality $2ab \le a^2 + b^2$.]
   b) What happens if you replace the Dirichlet boundary conditions by the Neumann boundary condition $\nabla u \cdot N = 0$ on the boundary (ends) of the string?
   c) Generalize part a) to a bounded region Ω in $\mathbb{R}^n$.

3. Two and three space dimensions

In higher space dimensions, the wave equation is $u_{tt} = c^2 \Delta u$. Thus, in two and three space dimensions

$$u_{tt} = c^2(u_{xx} + u_{yy}) \qquad \text{and} \qquad u_{tt} = c^2(u_{xx} + u_{yy} + u_{zz}). \tag{3.15}$$

Two dimensional waves on a drum head and waves on the surface of a lake are described by the first equation while sound and light waves are described by the second. Just as in the one dimensional case we can prescribe the initial position and initial velocity of the solution. For instance, in two space variables

$$\text{initial position} \qquad u(x,y,0) = f(x,y) \tag{3.16}$$

$$\text{initial velocity} \qquad u_t(x,y,0) = g(x,y). \tag{3.17}$$

a). Formulas for the solution in $\mathbb{R}^2$ and $\mathbb{R}^3$. There are standard formulas for the solution of the *initial value problem* (the term *Cauchy problem* is often called).

Technical Observation Let $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$. Say we want to solve

$$u_{tt} = \Delta u, \qquad \text{with} \quad u(x,0) = f(x) \quad \text{and} \quad u_t(x,0) = g(x). \tag{3.18}$$

Let $v(x,t)$ and $w(x,t)$, respectively, be the solutions of

$$v_{tt} = \Delta v, \qquad \text{with} \quad v(x,0) = 0 \quad \text{and} \quad v_t(x,0) = f(x). \tag{3.19}$$

and

$$w_{tt} = \Delta w, \qquad \text{with} \quad w(x,0) = 0 \quad \text{and} \quad w_t(x,0) = g(x). \tag{3.20}$$

Then v_t also satisfies the wave equation but with initial conditions $v_t(x,0) = f(x)$ and $v_{tt} = 0$. Thus the solution of (3.18) is $u(x,t) = v_t(x,t) + w(x,t)$. Since both (3.19) and (3.20) have zero initial position, one can find $u(x,t)$ after solving only problems like (3.20). This is utilized to obtain the following two formulas.

For the two (space) dimensional wave equation it is

$$u(x,y,t) = \frac{1}{2\pi c}\frac{\partial}{\partial t}\iint_{r\le ct} \frac{f(\xi,\eta)}{\sqrt{c^2t^2 - r^2}}\, d\xi\, d\eta + \frac{1}{2\pi c}\iint_{r\le ct} \frac{g(\xi,\eta)}{\sqrt{c^2t^2 - r^2}}\, d\xi\, d\eta, \tag{3.21}$$

where $r^2 = (x-\xi)^2 + (y-\eta)^2$.

In three (space) dimensions one has

$$u(x,y,z,t) = \frac{1}{4\pi c^2}\frac{\partial}{\partial t}\left(\iint_{r=ct} f(\xi,\eta,\zeta)\, dA\right) + \frac{1}{4\pi c^2 t}\iint_{r=ct} g(\xi,\eta,\zeta)\, dA, \tag{3.22}$$

where $r^2 = (x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2$ and dA is the element of surface area on the sphere centered at (x,y,z) with radius $r = ct$.

These are called *Kirchoff's formulas*. It is simplest first to obtain the formula in the three space dimensional case (3.22), and then obtain the two dimensional case (3.21) from the special three dimensional case where the initial data $f(x,y,z)$ and $g(x,y,z)$ are independent of z. This observation is called *Hadamard's method of descent*.

Exercises

1. Maxwell's equations for an electromagnetic field $E(x,t) = (E_1, E_2, E_3)$, $B(x,t) = (B_1, B_2, B_3)$ in a vacuum are

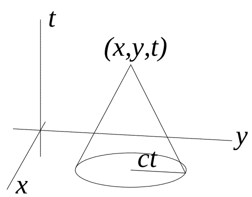
$$E_t = \operatorname{curl} B, \quad B_t = -\operatorname{curl} E, \quad \operatorname{div} B = 0, \quad \operatorname{div} E = 0.$$

Show that each of the components E_j and B_j satisfy the wave equation $u_{tt} = u_{xx}$. Also, show that if initially $\operatorname{div} B(x,0) = 0$ and

$\operatorname{div} E(x,0) = 0$, then $\operatorname{div} B(x,t) = 0$ and $\operatorname{div} E(x,t) = 0$ for all $t > 0$.

2. Let $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and consider the equation
$$\frac{\partial^2 u}{\partial t^2} = \sum_{j,k=1}^{n} a_{jk} \frac{\partial^2 u}{\partial x_j \partial x_k},$$
where the coefficients a_{jk} are constants and (without loss of generality — why?) $a_{kj} = a_{jk}$. If the matrix $A = (a_{jk})$ is positive definite, show there is a change of variable $x = Sy$, where S is an $n \times n$ invertible matrix, so that in these new coordinates the equation becomes the standard wave equation
$$\frac{\partial^2 u}{\partial t^2} = \sum_{\ell=1}^{n} \frac{\partial^2 u}{\partial^2 y_\ell}.$$

b). Domain of dependence and finite signal speed. As before, it is instructive to examine intersection of the domain of dependence with the plane $t = 0$, in other words, to determine the points x for which the initial data can influence the signal at a later time. In the two dimensional case (3.21), the intersection of the domain of dependence of the solution at (x_0, y_0, t_0) with the plane $t = 0$ is the entire disc $r \le ct_0$, while in the three dimensional case (3.22), the domain of dependence is only the sphere $r = ct_0$, *not* the solid ball $r \le ct_0$. Physically, this is interpreted to mean that two dimensional waves travel with a maximum speed c, but may move slower, while three dimensional waves always propagate with the exact speed c.



This difference in observed in daily life. If one drops a pebble into a calm pond, the waves (ripples) move outward from the center but ripples persist even after the initial wave has passed. On the other hand, an analogous light wave, such as a flash of light, moves outward as a sharply defined signal and does not persist after the initial wave has passed. Consequently, it is quite easy to transmit high fidelity waves in three dimensions — but not in two. Imagine the problems in attempting to communicate using something like Morse code with waves on the surface of a pond.

For the two space variable wave equation, the *characteristics* are the surfaces of all light cones $(x-\xi)^2 + (y-\eta)^2 = c^2 t^2$. In three space dimensions, the characteristics are the three dimensional light cones. They are the hypersurfaces in space-time with $(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2 = c^2 t^2$.

4. Energy and Causality

One can also give a different prove of results concerning the domain of dependence using an *energy method*. This technique is especially useful in more general situations where explicit formulas such as (3.21)–(3.22) are not available.

Let $x = (x_1, \ldots, x_n)$ and let $u(x,t)$ be a smooth solution of the n-dimensional wave equation

$$u_{tt} = c^2 \Delta u \qquad \text{where} \qquad \Delta u = u_{x_1 x_1} + \cdots + u_{x_n x_n}, \tag{3.23}$$

with initial data

$$u(x,0) = f(x), \qquad u_t(x,0) = g(x) \tag{3.24}$$

(Physicists often write the Laplacian, Δ, as ∇^2. Some mathematicians define Δ with a minus sign, so for them, in $\mathbb{R}^1$, $\Delta u = -u''$. Thus, one must be vigilant about the sign convention.)

a). Conservation of energy. Just as in the one dimensional case, we use the energy

$$E(t) = \tfrac{1}{2} \int_{\mathbb{R}^n} (u_t^2 + c^2 |\, \nabla u \,|^2)\, dx,$$

where we assume the solution is so small at infinity that this integral (and those below) converges. To prove conservation of energy, we show that $dE/dt = 0$, The computation is essentially identical to the one dimensional case we did above, only here we replace the integration by parts by the divergence theorem.

$$\frac{dE}{dt} = \int_{\mathbb{R}^n} (u_t u_{tt} + c^2 \nabla u \cdot \nabla u_t)\, dx = \int_{\mathbb{R}^n} u_t (u_{tt} - c^2 \Delta u)\, dx = 0.$$

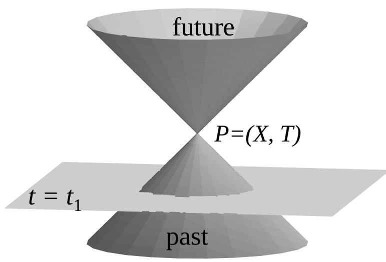
An immediate consequence of this is the uniqueness result: the wave equation (3.23) with initial conditions (3.24) has at most one solution. For if there were two solutions, v and w, then $u := v - w$ would be a solution of the wave equation with zero initial data, and hence zero initial energy. Since energy, $E(t)$, is conserved, $E(t) = 0$ for all time $t \ge 0$. Because the integrand in E is a sum of squares, then $u_t = 0$ and $\nabla u = 0$ for all $t \ge 0$. Thus $u(x,t) \equiv$const.. However $u(x,0) = 0$ so this constant can only be zero.

In two and three space dimensions this uniqueness also follows from the explicit formulas (3.21)–(3.22). However, the approach using energy also works when there are no explicit formulas.

b). Causality — using energy. Energy gives another approach to determine the domains of dependence and influence of the wave equation. Let $P = (X, T)$ be a point in space-time and let

$\mathcal{K}_P = \{ (x,t) \colon \|x - X\| \le c|\, t - T\,| \}$

be the *light cone* with apex P. This cone has two parts, that with $t > T$ is the *future light cone* while that with $t < T$ is the *past light cone*. In the two and three (space) dimensional case, from the explicit formulas for the solution we have seen that the value of the solution at P only depends on points in the past light cone, and can only influence the solution at points in the future light cone. Here we give another demonstration of this that does not rely on the earlier explicit formulas.



First, say $t_1 < T$ and let $D(t_1)$ be the intersection of $\mathcal{K}_P$ with the plane $t = t_1$. Define the "energy" function as

$$E(t) = \tfrac{1}{2} \int_{D(t)} (u_t^2 + c^2 |\, \nabla u\,|^2)\, dx.$$

Theorem 3.1. *If $u(x,t)$ is a solution of the wave equation, and if $t_1 < t_2 < T$, then*

$$\tfrac{1}{2} \int_{D(t_2)} (u_t^2 + c^2 |\, \nabla u\,|^2)\, dx \le \tfrac{1}{2} \int_{D(t_1)} (u_t^2 + c^2 |\, \nabla u\,|^2)\, dx,$$

that is, energy $E(t)$ is non-increasing for $t \le T$.
Consequently, if for some $t_1 < T$ we have $u(x,t_1) = 0$ and $u_t(x,t_1) = 0$ for all $x \in D(t_1)$, then $u(x,t) = 0$ for all points in the cone with $t_1 \le t \le T$.

Proof: We will show that $dE(t)/dt \le 0$ for $0 \le t \le T$. In $\mathbb{R}^n$ we use spherical coordinates centered at X, we have $dx = dr\, d\omega_r$, where $d\omega_r$ is the element of "area" on the $n-1$ sphere of radius r. Since the radius of the ball $D(t)$ is $c(T-t)$ we find that

$$E(t) = \tfrac{1}{2} \int_0^{c(T-t)} \left(\int_{S(r)} \left(u_t^2 + c^2 |\, \nabla u\,|^2\right)\, d\omega_r \right) dr,$$

where $S(r)$ is the $n-1$ sphere (in the plane) with radius r and centered at (X,t). Hence

$$\frac{dE}{dt} = \int_{D(t)} \left(u_t u_{tt} + c^2 \nabla u \cdot \nabla u_t\right) dx - \tfrac{c}{2} \int_{S(c(T-t)))} \left(u_t^2 + c^2 |\, \nabla u\,|^2\right) d\omega_{c(T-t)}.$$

Note that the integral on the right is just

$$\int_{\partial D(t)} \left(u_t^2 + c^2 |\, \nabla u\,|^2\right) dA,$$

where dA is the element of "area" on $\partial D(t)$. Since

$$\nabla u \cdot \nabla u_t = \nabla \cdot (u_t \nabla u) - u_t \Delta u,$$

then by the divergence theorem we have

$$\int_{\partial D(t)} \nabla u \cdot \nabla u_t = \int_{\partial D(t)} u_t (\nabla u \cdot \nu)\, dA - \int_{D(t)} u_t \Delta u\, dx,$$

where ν is the unit outer normal vector to $\partial D(t)$. Upon substituting into the formula for dE/dt, we find

$$E'(t) = \int_{D(t)} u_t (u_{tt} - c^2 \Delta u)\, dx + \tfrac{c}{2} \int_{\partial D(t)} \left[2 c u_t \nabla u \cdot \nu - \left(u_t^2 + c^2 |\, \nabla u\,|^2\right) \right] dA.$$

Next, we note that $u_{tt} = c^2 \Delta u$ so the first integral is zero. For the second term we use the standard inequality $2ab \le a^2 + b^2$ for any real a, b to obtain the estimate

$$|\, 2 c u_t \nabla u \cdot \nu\,| \le +2c |\, u_t \nabla u\,| \le u_t^2 + c^2 |\, \nabla u\,|^2.$$

Consequently, $E'(t) \le 0$. This completes the proof.

There are two immediate consequences of the energy inequality of this theorem.

Corollary 3.2 (uniqueness). . *There is at most one solution of the inhomogeneous wave equation $u_{tt} - c^2 \Delta u = f(x,t)$ with initial data* (3.24).

Corollary 3.3 (domain of influence). . *Let u be a solution of the initial value problem* (3.23)–(3.24). *If $u(x,0)$ and $u_t(x,0)$ are zero outside the ball $\{ \|x - X\| < \rho \}$, then for $t > 0$, the solution $u(x,t)$ is zero outside the forward light cone $\{ \|x - X\| < \rho + ct \}$, $t > 0$.*

Thus, for $t > 0$, the domain of influence of the ball $\{ \|x - X\| < \rho \}$ is contained in the cone $\{ \|x - X\| < \rho + ct \}$ in the sense that if one changes in the initial data only in this ball, then the solution can change only in the cone.

Exercise: Let $u(x,t)$ be a solution of the wave equation (3.14) for $x \in \mathbb{R}$. Use an energy argument to show that the solution u has the same domain of dependence and range of influence as in the special case where $a(x,t) = b(x,t) = 0$.

c). Mixed Initial-Boundary Value Problems. The above formulas (3.21), (3.22) were for waves in all of space. In the case of a vibrating membrane Ω, we must also impose boundary values on $\partial\Omega$, the boundary of Ω. Similarly, in the case of light or sound waves outside of Ω, we put boundary conditions on both $\partial\Omega$ and at "infinity" (this is sometimes referred to as an *exterior* problem, while a vibrating membrane is an *interior* problem. Just as for the vibrating string (...), two typical boundary conditions are

$$\begin{aligned} &u(x,t) = f(x,t) \quad \text{for} \quad x \in \partial\Omega \qquad \text{(Dirichlet conditions)} \\ &\frac{\partial u}{\partial \nu}(x,t) = g(x,t) \quad \text{for} \quad x \in \partial\Omega \qquad \text{(Neumann conditions)}, \end{aligned} \tag{3.25}$$

where $\partial/\partial\nu$ means the directional derivative in the outer normal direction to $\partial\Omega$. Of course this presumes that the boundary is smooth enough to have an outer normal direction. One also has situations where one of these conditions holds on part of the boundary and the other on another part. The vibrating string (3.10) is an example.
We now restrict our attention to waves in a bounded region Ω, such as a vibrating membrane, and use the method of separation of variables to solve the wave equation with homogeneous Dirichlet boundary conditions:

$$u_{tt} = \Delta u \tag{3.26}$$

$$u(x,t) = 0 \qquad \text{for} \qquad x \in \partial\Omega, \tag{3.26a}$$

$$u(x,0) = f(x), \qquad u_t(x,0) = g(x), \tag{3.26b}$$

where we have let $c = 1$. We seek special *standing wave solutions* in the form of a product

$$u(x,t) = W(x)T(t). \tag{3.27}$$

If one takes a sequence of photographs of such a solution at various times $t_1, t_2, \dots$, then you see the graph of $W(x)$ multiplied by the factor $T(t)$. The wave does not move horizontally, only up and down. In order to satisfy the boundary condition (3.26a) we need $W(x) = 0$ for $x \in \partial\Omega$. Substituting (3.27) into the wave equation we find that

$$\frac{\Delta W(x)}{W(x)} = \frac{T''(t)}{T(t)}.$$

Since the left side depends only on x while the right depends only on t, they must both be equal to a constant, say γ. Thus we obtain the two equations

$$T'' - \gamma T = 0 \qquad \Delta W = \gamma W. \tag{3.28}$$

We next observe that for a non-trivial solution (3.27) we must have $\gamma < 0$. To see this, multiply $\Delta W = \gamma W$ by W and integrate by parts

over Ω:

$$\gamma \int_\Omega W^2(x)\,dx = \int_\Omega W(x)\Delta W(x)\,dx = -\int_\Omega |\nabla W|^2\,dx,$$

where we use the boundary condition $W(x) = 0$ on $\partial\Omega$ to eliminate the boundary term. Solving this for γ we clearly see that $\gamma < 0$. In view of this, it will be convenient to write $-\gamma = \lambda$, so $\lambda > 0$. Then we have

$$-\Delta W = \lambda W \quad \text{in} \quad \Omega \qquad w = 0 \quad \text{on} \quad \partial\Omega \tag{3.29}$$

with

$$\lambda = \frac{\displaystyle\int_\Omega |\nabla W|^2\,dx}{\displaystyle\int_\Omega |W|^2\,dx}. \tag{3.30}$$

Thus λ is an eigenvalue of the operator, $-\Delta$ with corresponding eigenfunction $W(x)$. For a membrane Ω, these eigenvalues are essentially the squares of the various frequencies with which the membrane can vibrate and the eigenfunctions are the normal modes. It turns out that only a discrete sequence of eigenvalues $0 < \lambda_1 \le \lambda_2 \le \cdots$ are possible with $\lambda_j \to \infty$ as $j \to \infty$. Write the corresponding eigenfunctions as φ_j. From (3.28) the functions $T_j(t) = a_j \cos\sqrt{\lambda_j}t + b_j \sin\sqrt{\lambda_j}t$ so the standing wave solutions (3.27) are

$$u_j(x,t) = \left(a_j \cos\sqrt{\lambda_j}t + b_j \sin\sqrt{\lambda_j}t\right)\varphi_j(x), \tag{3.31}$$

where the a_j and b_j are arbitrary constants.
Next we seek the solution of (3.26), (3.26a), (3.26b) as a linear combination of these special standing wave solutions:

$$u(x,t) = \sum\left(a_j \cos\sqrt{\lambda_j}t + b_j \sin\sqrt{\lambda_j}t\right)\varphi_j(x). \tag{3.32}$$

To satisfy the initial conditions (3.26b) we choose the constants a_j and b_j so that

$$f(x) = \sum a_j\,\varphi_j(x) \qquad g(x) = \sum b_j\sqrt{\lambda_j}\,\varphi_j(x).$$

It is always possible to find these constants because the eigenfunctions φ_j are a complete orthonormal set on $L_2(\Omega)$. This series formally satisfies the differential equation, boundary conditions, and initial conditions. If f and g are sufficiently differentiable, then one can legitimately differentiate the above infinite series term-by-term to rigorously verify that $u(x,t)$ is an honest solution.

We can carry out these computations in only the simplest situations. The most basic is for a vibrating string of length π, so we let $\Omega = \{0 < x < \pi\}$. Then (3.29) is

$$W'' + \lambda W = 0, \qquad W(0) = 0, \quad W(\pi) = 0.$$

Thus $W(x) = A\cos\sqrt{\lambda}t + B\sin\sqrt{\lambda}t$. Now $W(0) = 0$ implies that $A = 0$. Then to obtain a non-trivial solution, $W(\pi) = 0$ implies that $\lambda_k = k^2$, $k = 1, 2, \ldots$ so the eigenfunctions are $\varphi_k(x) = \sin kx$, $k = 1, 2, \ldots$ and the series (3.31) is a classical Fourier series. The lowest eigenvalue, λ_1, is the fundamental tone of the string while the higher eigenvalues give the possible "overtones" or "harmonics".

Exercises:

1. In the above, investigate what happens if you replace the Dirichlet boundary condition $u(x,t) = 0$ for $x \in \partial\Omega$ by homogeneous Neumann boundary condition $\partial u/\partial N = 0$ for $x \in \partial\Omega$. Note here that $\lambda = 0$ is now an eigenvalue. What is the corresponding eigenfunction? Carry out the details for a vibrating string on the interval $0 < x < \pi$.

2. Find the motion $u(x,t)$ of a string $0 \le x \le \pi$ whose motion is damped:
$$u_{tt} + 3u_t = u_{xx},$$
with
$$u(x,0) = \sin 3x - 2\sin 5x, \quad u_t x, 0 = 0, \quad u(0,t) = u(\pi,t) = 0.$$

3. Prove the uniqueness of this solution of the problem (3.26) by an "energy" argument using (3.13).

5. Variational Characterization of the Lowest Eigenvalue

The formula (3.30) is essentially identical to the formula $\lambda = \langle x,\, Ax\rangle/\|x\|^2$ for the eigenvalues of a self-adjoint matrix A. A standard fact in linear algebra is that the lowest eigenvalue is given by $\lambda_1 = \min_{x\neq 0}\langle x,\, Ax\rangle/\|x\|^2$ (proof?). It is thus natural to surmise that the lowest eigenvalue of the Laplacian satisfies

$$\lambda_1 = \min \frac{\displaystyle\int_\Omega |\,\nabla\varphi\,|^2\,dx}{\displaystyle\int_\Omega |\,\varphi\,|^2\,dx}, \tag{3.33}$$

where the minimum is taken over all C^1 functions that satisfy the Dirichlet boundary condition $\varphi = 0$ on $\partial\Omega$. Assuming there is a function $\varphi \in C^2(\Omega)) \cap C^1(\overline{\Omega})$ that minimizes (3.30), we will show that it is an eigenfunction with lowest eigenvalue λ_1. To see this say such a φ minimizes the functional

$$J(v) = \frac{\displaystyle\int_\Omega |\,\nabla v\,|^2\,dx}{\displaystyle\int_\Omega |\,v\,|^2\,dx},$$

so $J(\varphi) \le J(v)$ for all $v \in C^1(\Omega)$ with $v = 0$ on $\partial\Omega$. Let $F(t) := J(\varphi + th)$ for any $h \in C^1(\Omega)$ with $h = 0$ on $\partial\Omega$ and all real t. Then $F(t)$ has its minimum at $t = 0$ so by elementary calculus, $F'(0) = 0$. By a straightforward computation, just as in the case of matrices,

$$F'(0) = 2\frac{\displaystyle\int_\Omega (\nabla\varphi\cdot\nabla h - \lambda_1\varphi h)\,dx}{\displaystyle\int_\Omega |\,\varphi\,|^2\,dx}.$$

We integrate the first term in the numerator by parts. There are no boundary terms since $h = 0$ on $\partial\Omega$. Thus

$$F'(0) = 2\frac{\displaystyle\int_\Omega [(-\Delta\varphi - \lambda_1\varphi)h]\,dx}{\displaystyle\int_\Omega |\,\varphi\,|^2\,dx}.$$

Since $F'(0) = 0$ for *all* of our h, we conclude the desired result:

$$-\Delta\varphi = \lambda_1\varphi.$$

Equation (3.33) is called the *variational characterization of the lowest eigenvalue*. There are analogous formulas for higher eigenvalues. Such formulas useful for computing numerical approximations to eigenvalues, and also to prove the existence of eigenvalues and eigenfunctions. The fraction in (3.33) is called the *Raleigh* (or *Raleigh-Ritz*) *quotient*.

Equation (3.33) implies the *Poincaré inequality*

$$\int_\Omega |\,\varphi\,|^2 \le c(\Omega)\int_\Omega |\,\nabla\varphi\,|^2\,dx \tag{3.34}$$

for all $\varphi \in C^1(\Omega)$ that vanish on $\partial\Omega$ (these are our admissible φ). Moreover, it asserts that $1/\lambda_1(\Omega)$ is the best value for the constant c.

It is instructive to give a direct proof of the Poincar'e inequality since it will give an estimate for the eigenvalue $\lambda_1(\Omega)$. Let V be a vector field on $\mathbb{R}^n$ (to be chosen later). For any of our admissible φ, by the divergence theorem

$$0 = \int_{\partial\Omega}\varphi^2 V\cdot N\,dA = \int_\Omega \operatorname{div}(\varphi^2 V)\,dx = \int_\Omega [\varphi^2\operatorname{div}V + \varphi\nabla\varphi\cdot V]\,dx,$$

where N is the unit outer normal vector field on $\partial\Omega$. Now pick V so that div $= 1$, say $V = (x_1 - \alpha, 0, \ldots, 0)$. Then picking the constant α appropriately, $|\,V\,| \le w/2$, where w is the width of Ω in the x_1 direction. Therefore, by the Schwarz inequality,

$$\int_\Omega [\varphi^2\,dx \le \frac{w}{2}\left[\int_\Omega \varphi^2\,dx\right]^{1/2}\left[\int_\Omega |\,\nabla\varphi\,|^2\,dx\right]^{1/2}.$$

Squaring both sides and canceling gives (3.34) with $c = (w/2)^2$, so $\lambda_1(\Omega) \ge w^2/4$.

Using the variational characterization(3.33), it is easy to prove a physically intuitive fact about vibrating membranes: *larger membranes have a lower fundamental frequency.* To prove this, say $\Omega \subset \Omega_+$ are bounded domains with corresponding lowest eigenvalues $\lambda_1(\Omega)$ and $\lambda_1(\Omega_+)$. Both of these eigenvalues are minima of the functional (3.33), the only difference being the class of functions for which the minimum is taken. Now every admissible function for the smaller domain Ω is zero on $\partial\Omega$ and hence can be extended to the larger domain by setting it to be zero outside Ω. It is now also an admissible function for the larger domain Ω_+. Therefore, for the larger domain the class of admissible functions for $J(v)$ is larger than for the smaller domain Ω. Hence its minimum $\lambda_1(\Omega_+)$ is no larger than $\lambda(\Omega)$.
Using similar reasoning, one can prove a number of related facts, and also get explicit estimates for eigenvalues. For instance, if we place $\Omega \subset \mathbb{R}^2$ in a rectangle Ω_+, since using Fourier series we can compute the eigenvalues for a rectangle, we get a lower bound for $\lambda(\Omega)$.

6. Smoothness of solutions

From the formula $u(x,t) = F(x-ct)+G(x+ct)$ for the solution of the one dimensional wave equation, since formally F and G can be any functions, it is clear that a solution of the wave equation need not be smooth (this is in contrast to the solutions of the Laplace equation, as we shall see later). In fact, in higher dimensions, even if the initial data (3.16) are smooth, the solution need not even be continuous. This can be seen intuitively for three space variables by choosing initial conditions on a sphere so that light rays are focuses at the origin at a later time. This is commonly done with a lens. To see this with formulas, notice that for any smooth $f \in C^\infty(\mathbb{R})$ the function

$$u(x,y,z,t) = \frac{f(r+ct)}{r}, \qquad \text{where} \quad r^2 = x^2+y^2+z^2,$$

formally satisfies the wave equation $u_{tt} = c^2\Delta u$. For small t it is a classical (that is, C^2,) solution even at $r=0$ if we pick a smooth function f so that $f(s) = 0$ for $|\,s\,| < 1$. The solution represents spherical waves coming to a *focus* at the origin. For such f both $u(x,y,z,0)$ and $u_t(x,y,z,0)$ are smooth everywhere., however, if say $f(2) \neq 0$, then at time $t = 2/c$ the solution $u(x,y,z,t)$ will blow-up at the origin. Nonetheless, one can make both physical and mathematical sense of this physically common situation. Since energy is conserved, the solution and its first derivatives are square-integrable. This can be used to define the concept of a *weak solution* of the wave equation. We take this up later (see ??).

7. The inhomogeneous equation. Duhamel's principle.

There are also formulas for the solution of the inhomogeneous wave equation

$$Lu := u_{tt} - c^2\Delta u = F(x,t). \tag{3.35}$$

The approach is analogous to Lagrange's method of variation of parameters, which gives a formula for the solution of an inhomogeneous equation such as $u''+u = F(t)$ in terms of solutions of the homogeneous equation. The method is called *Duhamel's principle.*
We illustrate it for the wave equation, seeking a solution of (3.35) with initial conditions

$$u(x,0) = 0 \quad u_t(x,0) = 0.$$

Since we are solving a differential equation, it is plausible to find a solution as an integral in the form

$$u(x,t) = \int_0^t v(x,t;s)\,ds \tag{3.36}$$

where the function $v(x,t;s)$, which depends on a parameter s, is to be found. This clearly already satisfies the initial condition $u(x,0) = 0$. Working formally, we have

$$u_t(x,t) = \int_0^t v_t(x,t;s)\,ds + v(x,t;t),$$

so $u_t(x,0) = 0$ implies $v(x,0;0) = 0$. In fact, we will further restrict v by requiring that $v(x,t;t) = 0$ for all $t \geq 0$. Then the formula for u_t simplifies and

$$u_{tt}(x,t) = \int_0^t v_{tt}(x,t;s)\,ds + v_t(x,t;s)\big|_{s=t}.$$

The similar formula for Δu is obvious. Substituting these into the wave equation (3.35) we want

$$F(x,t) = Lu(x,t) = \int_0^t Lv(x,t;s)\,ds + v_t(x,t;s)\big|_{s=t}.$$

This is evidently satisfied if $Lv = 0$ and $v_t(x,t;s)\big|_{s=t} = F(x,t)$ along with $v(x,t;t) = 0$ for all $t \geq 0$.
Because the coefficients in the wave equation do not depend on t, our results can be simplified a bit by writing $v(x,t;s) = w(x,t-s;s)$ so for each fixed s, the function $w(x,t;s)$ satisfies

$$w_{tt} = c^2\Delta w \qquad \text{with } w(x,0,;s) = 0 \text{ and } w_t(x,0;s) = F(x,s). \tag{3.37}$$

We can now find w by using our earlier formulas. For instance, in three space variables, from (3.22)

$$w(x,t;s) = \frac{1}{4\pi c^2 t} \iint_{\|\xi - x\| = ct} F(\xi, s)\, dA_\xi,$$

where dA_ξ is the element of surface area on the sphere centered at x with radius ct, that is, $\|\xi - x\| = ct$. Therefore from (3.36)

$$\begin{aligned} u(x,t) &= \frac{1}{4\pi c^2} \int_0^t \frac{1}{t-s} \iint_{\|\xi - x\| = c(t-s)} F(\xi, s)\, dA_\xi\, ds \\ &= \frac{1}{4\pi c} \int_0^t \iint_{\|\xi - x\| = c(t-s)} \frac{F(\xi, t - \|\xi - x\|/c)}{\|\xi - x\|}\, dA_\xi\, ds. \end{aligned}$$

But in spherical coordinates, the element of volume $d\xi = c dA_\xi ds$ so we finally obtain

$$u(x,t) = \frac{1}{4\pi c^2} \iiint_{\|\xi - x\| \le ct} \frac{F(\xi, t - \|\xi - x\|/c)}{\|\xi - x\|}\, d\xi. \tag{3.38}$$

Thus, to solve the inhomogeneous equation we integrate over backward cone $\|\xi - x\| \le ct$, which is exactly the domain of dependence of the point (x,t).

Exercises

1. Use Duhamel's principle to obtain a formula for the solution of
$$-u'' + k^2 u = f(x), \quad x \in \mathbb{R}, \quad \text{with} \quad u(0) = 0,\ u'(0) = 0.$$
Similarly, do this for $-u'' - k^2 u = f(x)$.

2. Use (3.21) to derive the analog of (3.38) for one and two space variables.

3. Let $x \in \mathbb{R}^n$.
   a) If function $w(x)$ depends only on the distance to the origin, $r = \|x\|$, show that
$$\Delta u = \frac{\partial^2 u}{\partial^2 r} + \frac{n-1}{r}\frac{\partial u}{\partial r}.$$
   b) Investigate solutions $u(x,t)$, $x \in \mathbb{R}^3$ of the wave equation $u_{tt} = \Delta u$ where $u(x,t) = v(r,t)$ depends only on r and t. For instance, let $v(r,t) := rw(r,t)$ and note that v satisfies a simpler equation. Use this to solve the wave equation in $\mathbb{R}^3$ where the initial data are radial functions:
$$v(r,0) = \varphi(r), \qquad v_t(r,0) = \psi(r).$$
[SUGGESTION: Extend both φ and ψ as even functions of r.]

Are there solutions of $w_t t(r,t) = \Delta(r,t)$ with the form $w(r,t) = h(r)g(r-t)$?

CHAPTER 4

The Heat Equation

1. Introduction

If we have a body Ω in $\mathbb{R}^n$, then under reasonable assumptions the differential equation

$$u_t = k\Delta u, \quad x \in \Omega \tag{4.1}$$

governs the temperature $u(x,t)$ at a point x at time t. Here $k > 0$, assumed constant in this example, describes the thermal conductivity of the body. It is large for copper and small for wood. By scaling x we can let $k = 1$. From experience in daily life, everyone has already done many experiments with heat flow. As we will see, to a surprising extent, the simple model of equation (4.1) embodies this intuition. This equation also describes diffusion.

2. Solution for $\mathbb{R}^n$

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a). Homogeneous equation. There are many approaches to get the formula for the solution of (4.1) in the special case where Ω is all of $\mathbb{R}^n$. Perhaps the most straightforward – but not the most elementary – is to use the Fourier transform.

$\mathbf{R}^1$ We first treat the one dimensional case of an infinite rod $-\infty < x < \infty$, so the problem is to solve the standard initial value problem

$$u_t = u_{xx} \qquad \text{with} \qquad u(x,0) = f(x). \tag{4.2}$$

Assuming a mild growth condition on f, say it is bounded and continuous, the solution is

$$u(x,t) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} f(s) e^{\frac{-(x-s)^2}{4t}}\, ds. \tag{4.3}$$

Before going further it is useful to make some observations based on this formula. First, it implies that if the initial temperature is non-negative but not identically zero, then the solution is positive everywhere, even for very small t. Thus, in contrast to the solution of the wave equation, heat conduction has an infinite signal speed. We also observe that even if f is, say, only piecewise continuous, the solution is smooth in both x and t for all $t > 0$. In fact, it has a power series in x that converges

for all x. Thus, the solution of the heat equation is a "smoothing operator".

To derive (4.3) for the moment we work formally and assume all integrals make sense. First take the Fourier transform of $u_t = u_{xx}$ with respect to the space variable x,

$$\hat{u}(\xi,t) := \int_{-\infty}^{\infty} u(x,t) e^{-ix\cdot\xi}\, dx.$$

Then from (4.27) in the Appendix to this chapter $\hat{u}(\xi,t)$ satisfies the ordinary differential equation

$$\hat{u}_t = -|\,\xi\,|^2 \hat{u} \qquad \text{with} \qquad \hat{u}(\xi,0) = \hat{f}(\xi)$$

in which ξ appears only as a parameter. It's solution is

$$\hat{u}(\xi,t) = e^{-|\,\xi\,|^2 t} \hat{f}(t).$$

Thus, by Fourier inversion (4.24) and the computation (4.23) we get the desired formula

$$\begin{aligned} u(x,t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-t|\,\xi\,|^2 + ix\cdot\xi} \hat{f}(\xi)\, d\xi \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} f(y) \Big(\int_{-\infty}^{\infty} e^{-t|\,\xi\,|^2 + i(x-y)\cdot\xi}\, d\xi \Big) dy \\ &= \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} f(y) e^{-\frac{|\,x-y\,|^2}{4t}}\, dy. \end{aligned}$$

This derivation was purely formal. Since the resulting formula may well hold under more general conditions than this derivation admits, instead of checking each step we verify directly that it solves the heat equation and satisfies the initial condition. By differentiating under the integral we immediately verify that it satisfies the heat equation $u_t = u_{xx}$ for all $t > 0$. Moreover u is a smooth function of x and t for all $t > 0$. It remains to show that $\lim_{t\downarrow 0} u(x,t) = f(x)$. This is a special case of the next lemma.

Let $\varphi_\lambda \in C(\mathbb{R})$ have the properties

(1) $\varphi_\lambda(x) \ge 0$,
(2) $\int_{\mathbb{R}} \varphi_\lambda(x)\, dx = 1$,
(3) For any $\delta > 0$, $\lim_{\lambda\downarrow 0} \int_{|\,y\,|\ge\delta} \varphi_\lambda(y)\, dy = 0$.

Let

$$f_\lambda(x) := \int_{\mathbb{R}} f(y) \varphi_\lambda(x-y)\, dy. \tag{4.4}$$

In applying this to the heat equation we will let

$$\varphi_\lambda(x) = \frac{1}{\sqrt{4\pi\lambda}} e^{-\frac{|\,x\,|^2}{4\lambda}}.$$

Comparing with (4.3) we see that $f_\lambda(x) = u(x,\lambda)$.

LEMMA 4.1. *If $f \in C(\mathbb{R})$ is bounded and φ as above, then $\lim_{\lambda\to 0} f_\lambda(x) = f(x)$, where the limit is uniform on compact subsets. Moreover, if φ is smooth, then so is f_λ.*

PROOF. To prove the uniform convergence in a compact interval $K \in \mathbb{R}$, given $\epsilon > 0$, use the uniform continuity of the continuous function f on a slightly larger interval K_1 to find $\delta > 0$ so that if $x \in K$ and $|w| < \delta$ with $(x-w) \in K_1$, then $|f(w) - f(x)| < \epsilon$. Also, say $|f(x)| \le M$. After the change of variable $x - y = z$ we get

$$\begin{aligned} f_\lambda(x) - f(x) &= \int_{\mathbb{R}} [f(y) - f(x)]\varphi_\lambda(x-y)\,dy = \int_{\mathbb{R}} [f(x-z) - f(x)]\varphi_\lambda(z)\,dz \\ &= \int_{|z|<\delta} [f(x-z) - f(x)]\varphi_\lambda(z)\,dz + \int_{|z|\ge\delta} [f(x-z) - f(x)]\varphi_\lambda(z)\,dz \end{aligned}$$

Thus,

$$|f_\lambda(x) - f(x)| < \epsilon + 2M \int_{|z|\ge\delta} \varphi_\lambda(z)\,dz$$

By Property 3) the last integral can be made arbitrarily small by choosing λ sufficiently small. Since the right hand side is independent of x (as long as $x \in K$), the convergence is uniform.

REMARK In $\mathbb{R}$ the *convolution* $f * g$ of f and g is defined as

$$(f * g)(x) = \int_{\mathbb{R}} f(y)g(x-y)\,dy.$$

The definition (4.4) defines f_λ as a convolution. Note that f is only continuous but g is smooth, then $f * g$ is smooth – assuming the integral exists. If the φ_λ are smooth, the above proof shows that on compact subsets we can uniformly approximated f by the smooth function f_λ. This technique of smoothing (or *mollifying*) a function is valuable. Weierstrass used (4.3) in his original proof of what we now call the *Weierstrass Approximation Theorem.*

Exercise:

a) Solve $u_t = u_{xx} + au$ for $x \in \mathbb{R}$, where $a = \text{const}$, with $u(x,0) = f(x)$. [SUGGESTION: Let $u(x,t) = \varphi(t)v(x,t)$, picking φ cleverly.
b) Solve $u_t = u_{xx} - bu_x$ for $x \in \mathbb{R}$, where $b = \text{const}$, with $u(x,0) = f(x)$. The term bu_x introduces *convection.* [SUGGESTION: Introduce a moving frame of reference by letting $y = x - bt$.]

$\mathbf{R}^n$ There is a similar formula for the solution of the heat equation for $x = (x_1, \ldots, x_n)$ in $\mathbb{R}^n$. In this case we seek a solution of

$$u_t = \Delta u \qquad \text{with initial temperature} \quad u(x,0) = f(x). \tag{4.5}$$

This solution is given by the formula

$$u(x,t) = \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} f(y) e^{-\frac{|x-y|^2}{4t}}\,dy. \tag{4.6}$$

To verify this, one uses the routine generalization of the above lemma.

Exercise: Use Fourier transforms to obtain (4.6).

Below we will use the maximum principle to show that with an essential boundedness assumption, the solution of (4.5) is unique.

b). Inhomogeneous equation. Using Duhamel's principle it is straightforward to obtain a formula for the solution of the inhomogeneous equation

$$u_t - \Delta u = F(x,t) \qquad \text{with} \quad u(x,0) = 0. \tag{4.7}$$

Seek u in the form

$$u(x,t) = \int_0^t v(x,t;s)\,ds.$$

Clearly $u(x,0) = 0$. Also, working formally,

$$u_t = \int_0^t v_t(x,t;s)\,ds + v(x,t;t).$$

with a similar formula for Δu. Consequently

$$u_t - \Delta u = \int_0^t [v_t - \Delta v]\,ds + v(x,t;t).$$

Since we want to solve $u_t - \Delta u = F$, it is natural to specify

$$v_t - \Delta v = 0 \qquad \text{with} \qquad v(x,t;s)|_{s=t} = F(x,t).$$

The function v is given by (4.6) except that we specify the initial temperature at $t = s$. Thus the desired solution $u(x,t)$ of (4.7) for $x \in \mathbb{R}^n$, $t > 0$, is

$$u(x,t) = \int_0^t \frac{1}{[4\pi(t-s)]^{n/2}} \int_{\mathbb{R}^n} f(y,s) e^{-\frac{|x-y|^2}{4(t-s)}}\,dy\,ds.$$

3. Initial-boundary value problems for a bounded region, part 1

To determine the temperature in a bounded region Ω, it is clear that we will need to know the initial temperature $u(x,0)$ and also something about the boundary. Two typical situations are that we might specify the temperature $u(x,t)$ at some boundary points x while ask that the boundary be insulated at other boundary points. As mentioned above, at a point where the boundary is insulated, the appropriate boundary condition is that the directional derivative in the outer normal direction is zero there: $\partial u/\partial N = 0$.

Thus, if we specify the temperature at all boundary points, we are asking to solve the heat equation with

(4.8) initial temperature $u(x,0) = f(x)$ for $x \in \Omega$

(4.9) boundary temperature $u(x,t) = g(x,t)$ for $x \in \partial\Omega$.

We call (4.9) a *Dirichlet boundary condition.*

The special case of the boundary condition $u(x,t) = g(x)$ means that the temperature at all boundary points does not depend on the time. Assuming this, here are two assertions that are intuitively clear.

- Say the initial and boundary temperatures are at most M. Then at any time in the future, the maximum temperature is at most M. This is called the *maximum principle.* We discuss it in the next section.
- Eventually, the temperature throughout the body tends to some "equilibrium temperature", $u(x,t) \to v(x)$, where $v(x)$ depends only on the boundary temperature, not on the initial temperature. This will be treated later in this chapter.

One test of the mathematical model is to prove these assertions from the data specified.

Above we specified Dirichlet boundary conditions. As an alternate, on some or all of the boundary of Ω one can prescribe the outer normal derivative, $\frac{\partial u}{\partial N} := N \cdot \nabla u$. This is the directional derivative in the direction of the outer normal:

$$\frac{\partial u}{\partial N} = g(x,t), \quad \text{for } x \in \partial\Omega. \tag{4.10}$$

This is called a *Neumann boundary condition.* The special case of an insulated boundary, so $\frac{\partial u}{\partial N} = 0$, arises frequently.
Mixed boundary conditions

$$u(x,t) + c(x,t)\frac{\partial u(x,t)}{\partial N} = g(x,t), \quad \text{for } x \in \partial\Omega$$

also arise occasionally.

4. Maximum Principle

To state the maximum principle we introduce some notation. If $\Omega \in \mathbb{R}^n$ is a bounded connected open set, for a fixed $T > 0$ let $\Omega_T := \Omega \times (0,T]$ so Ω is a cylinder in space-time. It's *parabolic boundary* is $P_T = \overline{\Omega}_T - \Omega_T$. This consists of the sides and bottom of the closed cylinder Ω_T. The maximum principle will be a consequence of the assertion

THEOREM 4.2. *In a bounded open set Ω, if the function $w(x,t)$ satisfies*

$$w_t - \Delta w \geq 0 \quad \text{for} \quad x \in \Omega_T, \tag{4.11}$$

and

(4.12)
$$w(x,0) \geq 0 \quad \text{for} \quad x \in \Omega \qquad \text{while} \quad w(x,t) \geq 0 \quad \text{for} \quad x \in \partial\Omega, \quad 0 \leq t \leq T$$

then either $w(x,t) > 0$ for all $x \in \Omega$, $0 < t \leq T$ or else $w(x,t) \equiv 0$ for all $x \in \overline{\Omega}$, $0 \leq t \leq T$.

For simplicity we prove only the weaker statement that $w(x,t) \geq 0$. First, to make the proof more transparent first assume that $w_t - \Delta w > 0$. Reasoning by contradiction, say $w(x,t) < 0$ somewhere in $S := \{\overline{\Omega} \times [0,T]\}$. Then it is negative at its absolute minimum at some interior point (x_0, t_0) with $x_0 \in \Omega_T$. But at this point, if $0 < t_0 < T$, we know that $w_t = 0$, while if $t_0 = T$ then $w_t \leq 0$. Moreover, by the second derivative test for a minimum we know that $\Delta w \geq 0$ at x_0. These facts contradict our assumption that $w_t - \Delta w > 0$.

Next, assume only that $w_t - \Delta w \geq 0$. We will use a limiting argument to prove that $w(x,t) \geq 0$. Again by contradiction, say $w(x_0, T) = m < 0$ at some interior point $x_0 \in \Omega$. Let $z(x,t) := w(x,t) - \epsilon |x - x_0|^2$. Pick $\epsilon > 0$ so small that $z(x,t) > m$ on P_T. Then z has its minimum at a point (x_1, t_1) where $x_1 \in \Omega_T$. Since $z_t - \Delta z > 0$, we can apply the reasoning of the above paragraph to obtain a contradiction.

COROLLARY 4.3 (Strong Maximum Principle). *In Ω_T assume the solution $u(x,t)$ of the heat equation is in C^2 for in $x \in \Omega$, C^1 for t in $(0,T]$ in t. Also assume that $u \in C(\overline{\Omega}_T)$. Then*

$$\max_{\overline{\Omega}_T} u(x,t) = \max_{P_T} u(x,t).$$

Moreover, if $u(x,t)$ attains its maximum at some point $(x_0, t_0) \in \Omega_T$, then u is constant throughout the cylinder $\overline{\Omega}_{t_0}$.

a). Applications of the maximum principle. Here are several typical consequences of the maximum principle.
Say

(4.13)
$$u_t - k\Delta u = F(x,t) \quad v_t - k\Delta v = G(x,t)$$

(4.14)
$$u(x,0) = f(x) \qquad v(x,0) = g(x) \qquad \text{for} \quad x \in \Omega$$

(4.15)
$$u(x,t) = \varphi(x,t) \qquad v(x,t) = \psi(x,t) \qquad \text{for} \quad x \in \partial\Omega, \; t > 0.$$

COROLLARY 4.4 (Comparison of solutions). *If*

$$F(x,t) \geq G(x,t), \; f(x) \geq g(x), \quad \textit{and} \quad \varphi(x,t) \geq \psi(x,t) \quad \textit{for all} \quad 0 \leq t \leq T,$$

then $u(x,T) \geq v(x,T)$, with strict inequality holding unless $F(x,t) \equiv G(x,t)$, $f(x) \equiv g(x)$, and $\varphi(x,t) \equiv \psi(x,t)$ for all $0 \leq t \leq T$.

Corollary 4.5 (Growth estimate). *Say* $|F(x,t)| \le M$, $|f(x)| \le c$ *and* $|\varphi(x,t)| \le c$. *Let* $w(x,t)$ *be a solution of*

(4.16)
$$w_t = \Delta w + 1 \quad \text{with} \quad w(x,0) = 0 \quad \text{in } \Omega \quad \text{and} \quad w(x,t) = 0,\ x \in \partial\Omega.$$

Then

$$|u(x,t)| \le c + M|w(x,t)| \quad \text{for} \quad t \ge 0. \tag{4.17}$$

Corollary 4.6 (Uniqueness). *There is at most one solution of* $u_t - k\Delta u = F(x,t)$ *with*

$$u(x,0) = f(x) \quad (x \in \Omega), \qquad \text{and} \qquad u(x,t) = \varphi(x,t) \quad (x \in \partial\Omega,\ t > 0).$$

Corollary 4.7 (Stability). *If the functions* F, f, *and* φ *are perturbed slightly, then the solution is perturbed only slightly. To be specific, say*

$$|F(x,t) - G(x,t)| < \alpha,\ |f(x) - g(x)| < \beta, \quad \text{and} \quad |\varphi - \psi| < \gamma \quad \text{for} \quad x \in \Omega,\ t \ge 0.$$

Then $|u(x,t) - v(x,t)| < \epsilon$ *for* $x \in \Omega$, $t \ge o$, *where* ϵ *is small if* α, β, *and* γ *are small.*

This is essentially just a restatement of (4.17) applied to $u - v$.

Exercises

1. Prove Corollary 4.

2. Prove Corollary 5.

3. Find an explicit estimate for the solution $w(x,t)$ in (4.16). The estimate will involve some property of Ω, such as its diameter.

4. Prove Corollary 6.

5. Prove Corollary 7.

b). Symmetry of solutions. Uniqueness is often the easiest approach to show that a solution possesses some symmetry. One example makes the ideas transparent. Let $\Omega \in \mathbb{R}^2$ be the rectangle $\{|x| < 1,\ 0 < y < 1\}$ and let $\gamma : \Omega \to \Omega$ be the reflection across the y-axis. Assume the initial and boundary temperatures are invariant under γ, so they are even functions of x. We claim the solution is also invariant under γ. This is obvious since both $u(x,t)$ and $v(x,t) := u(\gamma(x),t)$ are solutions of the heat equation with the same initial and boundary values.

c). Uniqueness in $\mathbb{R}^n$. If Ω is unbounded, such as an infinite rod $\{-\infty < x < \infty\}$, then the simple example $u(x,t) := 2t + x^2$ which satisfies the heat equation but whose maximum does not occur at $t = 0$ shows that the maximum principle fails unless it is modified. However, we can still use it as a tool. To illustrate, we'll prove a uniqueness theorem for the initial value problem (4.5) for the heat equation in all of $\mathbb{R}^n$. We prove uniqueness for this in the class of bounded solutions (one can weaken this to allow $u(x,t) \le \text{const} e^{\text{const}\,|x|^2}$, see [PW], p. 181, but there are examples of non-uniqueness if one allows faster growth). Say $u(x,t)$ satisfies the heat equation $u_t = \Delta u$ in all of $\mathbb{R}^n$ with $u(x,0) = 0$ and $|u(x,t)| \le M$. Inside the disk $\{|x| < a\}$ consider the comparison function $v(x,t;a) := M(|x|^2 + 2nt)/a^2$. Then v also satisfies the heat equation with

$$v(x,0;a) \ge 0 \quad \text{while} \quad v(x,t;a) \ge M \ge u(x,t) \quad \text{for} \quad |x| = a,\ t > 0.$$

Thus by the maximum principle $u(x,t) \le v(x,t;a)$ for $|x| \le a$. Fixing (x,t) but letting $a \to \infty$ we conclude that $u(x,t) \le 0$. Replacing u by $-u$ we then get $u(x,t) = 0$ for all $t \ge 0$.

Exercises

1. Let $u(x,t)$ be a bounded solution of the heat equation $u_t = u_{xx}$ with initial temperature $u(x,0) = f(x)$. If $f(x)$ is an odd function of $x \in \mathbb{R}$, show that the solution $u(x,t)$ is also an odd function of x.

2. Semi-infinite interval Solve the heat equation on a half-line: $0 < x < \infty$ with $u(x,0) = f(x)$ for $x \ge 0$ and the following conditions:
   a) $u(x,0) = f(x)$ for $x \ge 0$ and $u(0,t) = 0$ for $t \ge 0$. [Suggestion: Extend $f(x)$ cleverly to $x < 0$.]
   b) $u(x,0) = 0$ and $u(0,t) = g(t)$. [Suggestion: Let $v(x,t) = u(x,t) - g(t)$.]
   c) $u(x,0) = f(x)$ for $x \ge 0$ and $u(0,t) = g(t)$ for $t \ge 0$.

5. Initial-boundary value problems for a bounded region, part 2

a). Using separation of variables. We seek special solutions of the heat equation in a bounded region Ω with zero Dirichlet boundary conditions:

(4.18)
$$u_t = \Delta u \quad \text{for } x \in \Omega, \quad \text{with} \quad u(x,0) = f(x) \quad \text{and} \quad u(x,t) = 0\ x \in \partial\Omega.$$

Because regions Ω are rarely simple, one can almost never fill-in many details, yet even working crudely one can get useful information. Just as for the wave equation, one can use separation of variables to seek special solutions $u(x,t) = v(x)T(t)$ with $v = 0$ on the boundary of Ω. As before, v must be an eigenfunction v_k of the Laplacian, with eigenvalue λ_k, that is, $-\Delta v_k = \lambda_v v_k$. We may assume the eigenfunctions are orthonormal. Then $T_k(t) = e^{-\lambda_k t}$ so the special solutions are $u_k(x,t) =$

$v_k(x)e^{-\lambda_k t}$. We build the general solution as a linear combination:

$$u(x,t) = \sum a_k v_k(x) e^{-\lambda_k t}, \tag{4.19}$$

where the a_k are found using the initial condition

$$f(x) = u(x,0) = \sum a_k v_k(x) \qquad \text{so} \quad a_k = \langle f,\, v_k\rangle.$$

Consequently, working formally,

(4.20)

$$u(x,t) = \sum \langle f,\, v_k\rangle v_k(x) e^{-\lambda_k t} = \sum \int f(y) v_k(y) v_k(x) e^{-\lambda_k t} dy$$

(4.21)

$$= \int f(y) G(x,y) dy, \qquad \text{where} \quad G(x,y,t) = \sum v_k(y) v_k(x) e^{-\lambda_k t}.$$

Here $G(x,y,t)$ is called *Green's function* for the problem. Because the eigenvalues, λ_k are all positive, it is clear from (4.19) that $u(x,t) \to 0$ as t tends to infinity. This should agree with your physical intuition. The lowest eigenvalue, λ_1, determines the decay rate.

Exercise: Repeat this using homogeneous Neumann boundary conditions $\partial u/\partial N = 0$ on the boundary. What can you say about $\lim_{t\to\infty} u(x,t)$?

b). Another approach. Using techniques similar to the energy methods we used for the wave equation, we can also obtain information about solutions of the heat equation. These are reasonable exercises.

Exercises

1. Let $u(x,t)$ be a solution of the heat equation $u_t = \Delta u$ in Ω with $u = 0$ on $\partial\Omega$. Define

$$H(t) := \tfrac{1}{2} \int_\Omega u^2(x,t)\, dx.$$

Show that $dH/dt \le 0$. Then use this to prove a uniqueness theorem.

2. [Improvement of the previous Exercise] Use the variational characterization of λ_1 (see our discussion of the wave equation) to show that

$$\frac{dH}{dt} \le -\lambda_1 H(t).$$

Then use this to show that $H(t) \le ce^{\lambda_1 t}$ for some constant c. This proof of decay is independent of the previous version that used separation of variables.

3. Let $u(x,t)$ be a solution of the heat equation $u_t = \Delta u$ in Ω with homogeneous Neumann boundary conditions, $\partial u/\partial N = 0$ on $\partial\Omega$, so the boundary is insulated. Show that $Q(t) := \int_\Omega u(x,t)\, dx =$ constant.

4. Find a modified version of Exercises 1-2 above for the case of homogeneous Neumann boundary conditions.

6. Appendix: The Fourier transform

To derive the standard formula for the solution of the heat equation of an infinite rod, we used the Fourier transform. Here is a brief summary of basic facts about the Fourier transform. If $u \in L^1(\mathbb{R}^n)$, its *Fourier transform* $\hat{u}(\xi)$ is defined as

$$\hat{u}(\xi) := \int_{\mathbb{R}^n} u(x) e^{-ix\cdot\xi}\, dx, \tag{4.22}$$

It is evident that $|\,\hat{u}(\xi)\,| \le \|u\|_{L^1(\mathbb{R}^n)}$.

a). A special integral.

LEMMA 4.8. *Let A be a real $n\times n$ positive definite symmetric matrix and b a complex vector. Then*

$$I := \int_{\mathbb{R}^n} e^{-x\cdot Ax + b\cdot x}\, dx = \left(\frac{\pi^n}{\det A}\right)^{\frac{1}{2}} e^{b\cdot A^{-1}b/4}, \tag{4.23}$$

where $b\cdot x$ is the usual inner product in $\mathbb{R}^n$.

PROOF. Since A is positive definite, it has a positive definite square root P, $P^2 = A$; this is obvious in a basis in which A is diagonalized. Make the (real) change of variables $y = Px$ in the above integral. Then we have $dy = (\det P)\, dx = (\det A)^{1/2}\, dx$ and with $\gamma := \frac{1}{2}P^{-1}b$, by completing the square

$$x\cdot Ax - b\cdot x = |\,y\,|^2 - 2\gamma\cdot y = |\,y-\gamma\,|^2 - \tfrac{1}{4} b\cdot A^{-1}b.$$

Let $z := y - \gamma$ and $c := Im\{\gamma\}$. Then the above integral, I becomes

$$I = \frac{e^{b\cdot A^{-1}b/4}}{(\det A)^{1/2}} \prod_{j=1}^n \left(\int_{-\infty - ic_j}^{\infty - ic_j} e^{-z_j^2}\, dz_j \right).$$

To complete the computation we need to evaluate the complex integrals on the right. In the complex ζ plane, integrate around the rectangle with vertices at $(\pm R, \pm R - iq)$, where R and q are real, and let $R\to\infty$ to conclude that

$$\int_{-\infty - iq}^{\infty - iq} e^{-\zeta^2}\, d\zeta = \int_{-\infty}^{\infty} e^{-\zeta^2}\, d\zeta = \sqrt{\pi}.$$

Combined with the above formula for I this gives the desired formula.

We use this to compute the Fourier transform of $\psi(x) = e^{-|x|^2/2}$:

$$\hat{\psi}(\xi) = \int_{R^n} e^{-|x|^2/2 - ix\cdot\xi}\, dx = (2\pi)^{n/2} e^{-|\xi|^2/2}. \tag{4.24}$$

Thus $e^{-|x|^2/2}$ is an eigenfunction of the Fourier transform operator.

b). Inversion of the Fourier transform. The formula

$$u(x) = \frac{1}{(2\pi)^n} \int_{-\infty}^{\infty} \hat{u}(\xi) e^{ix\cdot\xi}\, d\xi \tag{4.25}$$

shows how to recover a function from its Fourier transform. To prove this, say $u \in L^1(\mathbb{R}^n)$ is a bounded function and pick some ψ so that both ψ and $\hat{\psi}$ are bounded and in $L^1(\mathbb{R}^n)$ (below we make the specific choice $\psi(x) = e^{-|x|^2/2}$). Use the notation $\psi_\lambda(\xi) := \psi(\lambda\xi)$. Then by an easy computation its Fourier transform is $\hat{\psi}_\lambda(y) = \lambda^{-n}\hat{\psi}(y/\lambda)$. Now

(4.26)
$$\begin{aligned}
\int_{-\infty}^{\infty} \hat{u}(\xi)\psi_\lambda(\xi)e^{ix\cdot\xi}\, d\xi &= \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} u(y) e^{-iy\cdot\xi}\, dy \right) \psi_\lambda(\xi) e^{ix\cdot\xi}\, d\xi \\
&= \int_{\mathbb{R}^n} u(y) \left(\int_{\mathbb{R}^n} \psi_\lambda(\xi) e^{-i(y-x)\cdot\xi}\, d\xi \right) dy \\
&= \int_{\mathbb{R}^n} u(y)\hat{\psi}_\lambda(y-x)\, dy = \int_{\mathbb{R}^n} u(x+t\lambda)\hat{\psi}(t)\, dt.
\end{aligned}$$

In this computation we were permitted to interchange the orders of integration since the integrals all converge absolutely. Because u is bounded, by the dominated convergence theorem we can let $\lambda \to 0$ to obtain the identity

$$\psi(0) \int_{\mathbb{R}^n} \hat{u}(\xi) e^{ix\cdot\xi}\, d\xi = u(x) \int_{\mathbb{R}^n} \hat{\psi}(t)\, dt.$$

Choosing $\psi(x) = e^{-|x|^2/2}$ and using (4.24) gives the desired Fourier inversion formula, at least for bounded functions $u \in L^1(\mathbb{R})$.

c). Fourier transform of the derivative. One reason the Fourier transform is so useful when discussing linear differential equations with constant coefficients is that the Fourier transform changes differentiation into multiplication by a polynomial. This is easily seen by integrating by parts

$$\widehat{\partial_j u}(\xi) = \int_{-\infty}^{\infty} \partial_j u(x) e^{-ix\cdot\xi}\, dx = i\xi_j \hat{u}(\xi). \tag{4.27}$$

In particular, $\widehat{(\Delta u)}(\xi) = -|\xi|^2 \hat{u}(\xi)$. so for any integer $k \geq 0$

$$(1 + |\xi|^2)^k \hat{u}(\xi) = [\widehat{(1-\Delta)^k u}](\xi).$$

Therefore, if $u \in \mathbb{C}^\infty(\mathbb{R}^n)$, then for any integer $k \geq 0$ there is a constant c depending on u and k so that

$$|\hat{u}(\xi)| \leq \frac{c}{(1+|\xi|^2)^k}. \tag{4.28}$$

It is useful to compare this to the result in Chapter 2 concerning the decay of Fourier coefficients of smooth functions. They are essentially identical.

CHAPTER 5

The Laplace Equation

1. Introduction

As we saw in the previous chapter, if $v(x,t)$ is a solution of the heat equation and if that solution converges to an "equilibrium" state $u(x)$, then u is a solution of the Laplace equation:

$$\Delta u = 0. \tag{5.1}$$

These are called *harmonic functions*.

Harmonic functions are invariant under both *translations*: $x \to x - a$, *orthogonal transformations*: $x \to Rx$, and *scalings*: $x \to \lambda x$.
Although we will not exploit it here, less obvious is that is the behavior under *inversions in the unit sphere*: $x \to x^* = x/|\,x\,|^2$. Note that x^* is on the same ray from the origin as x and $|\,x^*\,||\,x\,| = 1$. Given a domain Ω, let Ω^* be its image under this inversion. For instance, the inversion of the unit ball $|\,x\,| < 1$ is the exterior of this same ball. In dimension two, harmonic functions are invariant under inversions. For higher dimensions, define the *Kelvin transform* by

$$\mathcal{K}(u)(x) := \frac{u(x^*)}{|\,x\,|^{n-2}}.$$

If $u(x)$ is harmonic in Ω, then $\mathcal{K}(u)$ is harmonic in Ω^*. This follows from the identity $\Delta(\mathcal{K}(u)) = \mathcal{K}(|\,x\,|^4 \Delta u)$ which is most easily proved first for homogeneous polynomials and then use that one can approximate any $u \in C^2$ (in the C^2 norm) by a polynomial.

In this chapter we will also briefly discuss both harmonic functions and solutions of the inhomogeneous equation

$$-\Delta u = f(x),$$

which is called the *Poisson Equation*.

A useful reference for this chapter is the first part of the book
Axler, S., Bourdin, P., and Ramey, *Harmonic Function Theory*, accessible at
`http://www.axler.net/HFT.pdf`.

2. Poisson Equation in $\mathbb{R}^n$

We first seek a particular solution of the Poisson Equation in $\mathbb{R}^n$. For this, we look for a solution of the very special equation

$$-\Delta\Phi = \delta_0, \tag{5.2}$$

where $\delta_0(x)$ is the *Dirac delta measure* concentrated at the origin.

Since $\delta_0(x) = 0$ except at $x = 0$ and since the Laplacian is invariant under orthogonal transformations, it is plausible to seek a solution $\Phi(x)$ of (5.2) as a function depending only on the radial direction $r = |\,x\,|$, so $\Phi(x) = v(r)$ is harmonic away from the origin. By the chain rule,

$$\Delta v(r) = \frac{d^2v}{dr^2} + \frac{n-1}{r}\frac{dv}{dr}.$$

Thus we seek solutions of the ordinary differential equation

$$v'' + \frac{n-1}{r}v' = 0.$$

This is straightforward and gives

$$v(r) = \begin{cases} a\log r + b & \text{for } \ n = 2, \\ \frac{a}{r^{n-2}} + b & \text{for } \ n \geq 3. \end{cases}$$

To get a solution of (5.2), one lets $b = 0$ and picks a appropriately to define

$$\Phi(x) = \begin{cases} -\frac{1}{2\pi}\log|\,x\,|, & \text{for } n = 2 \\ \frac{1}{n(n-2)\alpha_n |\,x\,|^{n-2}} & \text{for } n \geq 3, \end{cases} \tag{5.3}$$

where α_n is the volume of the unit ball $B(0,1) = \{x \in \mathbb{R}^n |\,|\,x\,| < 1\}$, so $n\alpha_n$ is the area of the unit sphere $S^{n-1} = \partial B(0,1)$. This function $\Phi(x)$ is called the *fundamental solution* of the Laplacian.
Since $-\Delta\Phi = \delta_0$, we guess that a solution of $-\Delta u = f$ is given by

$$u(x) = \int_{\mathbb{R}^n} f(y)\Phi(x-y)\,dy. \tag{5.4}$$

To verify this, it is tempting to take the Laplacian of both sides, but since we believe $\Delta\Phi(x-y) = \delta_x$, which is highly singular at $y = x$, we must proceed more carefully – and need to assume some smoothness for f (assuming $f \in \mathbb{C}^1$ is more than enough). The details of this verification are in many books.

3. Mean value property

The *mean value property* for a harmonic function u states that $u(x)$ is the average of its values on any sphere centered at x

$$u(x) = \frac{1}{\text{Area}\,(\partial B(x,r))}\int_{\partial B(x,r)} u(y)\,dA_y. \tag{5.5}$$

To prove this, let $d\omega$ be the element of area on the unit sphere; then on $\partial B(x_0, r)$ we have $dA = r^{n-1} d\omega$,

(5.6)

$$0 = \int_{B(x,r)} \Delta u(y)\, dy = \int_{\partial B(x,r)} \frac{\partial u}{\partial N}\, dA = r^{n-1} \int_{|\xi|=1} \frac{\partial u(x + r\xi)}{\partial r}\, d\omega_\xi$$

$$= r^{n-1} \frac{d}{dr} \left(\int_{|\xi|=1} u(x + r\xi)\, d\omega_\xi \right). \tag{5.7}$$

Thus the last integral on the right is independent of r. Letting $r \to 0$ we obtain

$$\int_{|\xi|=1} u(x + r\xi)\, d\omega_\xi = \text{Area}\,(\partial B(0,1)) u(x),$$

which is (5.5).

The *solid mean value property* is

$$u(x) = \frac{1}{\text{Vol}\,(B(x,r))} \int_{B(x,r)} u(y)\, dy.$$

It follows from the mean value property for spheres by simply multiplying both sides by $\text{Area}\,(\partial B(x,r))$ and integrating with respect to r.

The *maximum principle* is an easy consequence. It asserts that if u is harmonic in a connected bounded open set Ω and continuous in $\bar{\Omega}$, then

$$\max_{\bar{\Omega}} u(x) \le \max_{\partial\Omega} u(x).$$

Moreover, if u attains its maximum at an interior point, then $u \equiv$ constant in Ω.

Since u is a continuous function on the compact set $\bar{\Omega}$ and hence attains its maximum somewhere, we need only prove the second assertion. Say u attains its maximum at an interior point $x_0 \in \Omega$ and say $u(x_0) = M$. Let $Q = \{x \in \Omega | u(x) = M\}$. Since u is continuous then Q is closed. By the mean value property, Q is open. Since Q is not empty and Ω is connected, it follows that $Q = \Omega$.

There is an obvious *minimum principle* which follows by replacing u by $-u$.

Uniqueness for the Dirichlet problem in a bounded connected region Ω

$$\Delta u = f \text{ in } \Omega, \quad \text{with} \quad u = \varphi \text{ on } \partial\Omega$$

is easy. We need only prove that if $\Delta u = 0$ in Ω and $u = 0$ on $\partial\Omega$, then $u \equiv 0$. But if u is not identically zero, it is either positive or negative

somewhere inside Ω and thus attains its maximum or minimum at an interior point. This contradicts the maximum principle.

One can use an "energy" approach to give an alternative proof. By the divergence theorem, if u is harmonic in Ω and zero on the boundary, then

$$0 = \int_\Omega u \Delta u\, dx = -\int_\Omega |\nabla u|^2\, dx$$

so $u \equiv$ constant. But since $u = 0$ on the boundary, $u \equiv 0$.

Exercises:

1. Show that this second proof also works with Neumann boundary conditions $\partial u / \partial N = 0$, except that with these boundary conditions we can only conclude that $u \equiv$ constant. Indeed, if u is any solution, then so is $u +$ const.

2. If in a bounded domain say $\Delta u = 0$ with $u = f$ on the boundary while $\Delta v = 0$ with $v = g$ on the boundary. If $f < g$ what can you conclude? Proof?

3. If u satisfies $-\Delta u \ge 0$, show that the average of u on any sphere is at least its value at the center of the sphere. Use this to conclude that if $u \ge 0$ on the boundary of a bounded domain Ω, then $u \ge 0$ throughout Ω.

4. In a domain $\Omega \subset \mathbb{R}^n$ let $u(x)$ be a solution of $-\Delta u + a(x) u = 0$, where $a(x) > 0$.
   a) Show there is no point where u has a positive local maxima (or negative minima).
   b) In a bounded domain, show that there is at most one solution of the Dirichlet problem
   $$-\Delta + a(x) u = F(x) \text{ in } \Omega \quad \text{with} \quad u = \varphi \text{ on } \partial\Omega.$$
   [Give two different proofs, one using part a), the other using "Energy."]

5. In a domain $\Omega \subset \mathbb{R}^n$ let the vector $u(x)$ be a solution of the system of equations $-\Delta u + A(x) u = 0$ with $u = 0$ on the boundary. Here $A(x)$ is a symmetric matrix and Δu means apply Δ to each component of u; a useful special case is the system of ordinary differential equations $-u'' + A(x) u = 0$. Assume $A(x)$ is a positive definite matrix, show that $u \equiv 0$. Also, give an example showing that if one drops the assumption that $A(x)$ is positive definite, then there may be non-trivial solutions. SUGGESTION: As just above, there are two distinct approaches, both useful:
   i). Use energy methods directly.

ii). Let $\varphi(x) = |\nabla u(x)|^2$ and apply the scalar maximum principle to $\Delta\varphi$.

6. On the torus, $\mathbb{T}^2$, let u be a solution of
$$-\Delta u = 1 - h(x)e^u,$$
where $h \in \mathbb{C}(\mathbb{T}^2)$, so in particular, h is a periodic function of its variables. If $0 < a \le h(x) \le b$, find upper and lower bounds for u in terms of a and b.

7. Let $(a_{ij}(x))$ be a positive definite $n\times n$ matrix for $x \in \mathbb{R}^n$. Assume $u(x) \in C^2$ satisfies
$$-\sum_{i,j=1}^{n} a_{ij}\frac{\partial^2 u}{\partial x_i \partial x_j} + c(x)u = 0,$$
where $c(x) > 0$.
   a) Show that u cannot have a local positive maximum. Also show that u cannot have a local negative minimum.
   b) If a function u satisfies the above equation in a bounded region $\mathcal{D} \in \mathbb{R}^n$ and is zero on the boundary of the region, show that $u(x)$ is zero throughout the region.

Using only the solid mean value property one can prove a *weak Harnack inequality*. Assume the harmonic function $u \ge 0$ in the ball $|x - x_0| \le R$. Then for any point x_1 in this ball

$$0 \le u(x_1) \le \frac{R^n}{(R - |x_1|)^n} u(x_0). \tag{5.8}$$

To prove this, since the ball $B(x_1, R - |x_1|) \subset B(x_0, R)$ we use the solid mean value property in $B(x_1, R - |x_1|)$, the assumption that $u \ge 0$, and the mean value property a second time to find

$$\begin{aligned} u(x_1) &= \frac{1}{\mathrm{Vol}\,(B(x_1, R - |x_1|)} \int_{B(x_1,\, R - |x_1|)} u(x)\,dx \\ &\le \frac{1}{\mathrm{Vol}\,(B(x_1,\, R - |x_1|)} \int_{B(x_0,R)} u(x)\,dx \\ &= \frac{\mathrm{Vol}\,(B(x_0, R))}{\mathrm{Vol}\,(B(x_1,\, R - |x_1|)} u(x_0) = \frac{R^n}{(R - |x_1|)^n} u(x_0). \end{aligned}$$

Inequality (5.8) quickly implies a *Liouville theorem*: If u is harmonic on all of $\mathbb{R}^n$ and $u \ge 0$, then $u(x) \equiv \text{const}$. Indeed, by letting $R \to \infty$ in (5.8) we find that $u(x_1) \le u(x_0)$ for any two points x_0, x_1. Interchanging the roles of these points we see that $u(x_0) \le u(x_1)$. Consequently $u(x_1) = u(x_0)$.

4. Poisson formula for a ball

There are very few domains Ω for which one has an explicit formula for the solution $u(x)$of the Dirichlet problem

$$\Delta u = 0 \text{ in } \Omega \quad \text{with } u = f \text{ on } \partial\Omega. \tag{5.9}$$

A valuable special case is if Ω is a ball $B(0, R) \subset \mathbb{R}^n$. Then the solution is given by the *Poisson formula*

$$u(x) = \frac{R^2 - |x|^2}{n\alpha_n R} \int_{\partial B(0,R)} \frac{f(y)}{|x - y|^n}\, dA(y). \tag{5.10}$$

The function
$$P(x, y) = \frac{R^2 - |x|^2}{n\alpha_n R|x - y|^n}$$
is called the *Poisson kernel*.

There are several ways to derive (5.10). If $n = 2$ one can use separation of variables in polar coordinates. Other techniques are needed in higher dimensions. The details are carried out in all standard texts.

The mean value property is the special case of (5.10) where $x = 0$.

One easy, yet important, consequence of the Poisson formula is that if a function u is harmonic inside a domain Ω, then it is smooth (C^∞) there. To prove this near a point x consider a small ball $B(x_0, R) \subset \Omega$ containing x and use (5.10) to obtain a formula for u in terms of its values on the boundary of the ball:
$$u(x) = \frac{R^2 - |x|^2}{n\alpha_n R} \int_{\partial B(x_0,R)} \frac{u(y)}{|x - y|^n}\, dA(y).$$
Since y is on the boundary of the ball and x is an interior point, one can repeatedly differentiate under the integral sign as often as one wishes. By a more careful examination, one can even see that $u(x)$ is real analytic, that is, locally it has a convergent power series expansion.

Another consequence of the Poisson formula is a *removable singularity* assertion. Say $u(x)$ is harmonic in the punctured disk $0 < |x - x_0| \le R$ and bounded in the disk $|x - x_0| \le R$. Then u and be extended uniquely to an harmonic function in the whole disk $|x - x_0| \le R$.

The proof goes as follows. Without loss of generality we may assume that $x_0 = 0$. Using the Poisson formula, in $\{|x| \le R\}$ we can find a harmonic function v with $v(x) = u(x)$ for $|x| = R$. Given any $\epsilon > 0$ let
$$w(x) = u(x) - v(x) - \epsilon[\Phi(x) - \Phi(R)],$$
where $\Phi(x)$ is the fundamental solution (5.3) of the Laplacian. Clearly $w(x) = 0$ on $|x| = R$ while, since $u(x)$ is bounded, then $w(x) < 0$ on $|x| = \delta$ for $\delta > 0$ sufficiently small. Consequently $w(x) \le 0$ in

$\delta < |x| \le R$; equivalently, $u(x) \le v(x) + \epsilon[\Phi(x) - \Phi(R)]$. Since ϵ is arbitrary, $u(x) \le v(x)$ in the annular region $\delta < |x| \le R$.
Similarly, by considering

$$w(x) = u(x) - v(x) + \epsilon[\Phi(x) - \Phi(R)],$$

we deduce that $u(x) \ge v(x)$ in this same annular region. Consequently, $u(x) = v(x)$ in this region. Since we can make δ arbitrarily small and since $v(x)$ is continuous at the origin, if we define $u(0) = v(0)$, the function $u(x)$ is harmonic throughout the disk $|x| \le R$.

Remark: In this proof, we could even have allowed $u(x)$ to blow-up near the origin, as long as it blows-up slower than the fundamental solution $\Phi(x)$. To state it we use "little o" notation:

$$g(s) = o(h(s)) \text{ as } s \to s_0 \quad \text{means} \quad \lim_{s\to s_0} g(s)/h(s) = 0.$$

For example $x^2 = o(x)$ as $x \to 0$. In this notation, the precise assumption needed on $u(x)$ for a removable singularity is

$$u(x) = o(\Phi(x)) \quad \text{as} \quad x \to 0.$$

Exercises:

1. Use separation of variables in polar coordinates to obtain the Poisson formula for the unit disk in $\mathbb{R}^2$.

2. Use separation of variables in polar coordinates to solve the Dirichlet problem for the annulus $0 < a^2 < x^2 + y^2 < 1$ in $\mathbb{R}^2$.

3. Let u_k be a sequence of harmonic functions that converge uniformly to some function $u(x)$ in a domain Ω. Show that u is also harmonic.

4. [Harnack inequality] Let $u(x) \ge 0$ be harmonic in the ball $B(0, R)$. Use the Poisson formula to show that

$$R^{n-2}\frac{R - |x|}{(R + |x|)^{n-1}}u(0) \le u(x) \le R^{n-2}\frac{R + |x|}{(R - |x|)^{n-1}}u(0).$$

5. Existence and regularity for $-\Delta u + u = f$ on $\mathbb{T}^n$

We will use Fourier series to solve $-\Delta u + u = f$ on the torus T^n. [See the last section of Chapter 2 for basics on Fourier series.] This equation is a bit simpler than $\Delta u = f$ since the homogeneous equation $\Delta u = 0$ has the non-trivial solution u =const. Despite that the solution of $-\Delta u + u = g$ will be an infinite series, the results and insight gained are valuable.

a). $-\Delta u + u = f$ on $\mathbb{T}^n$. If f has a Fourier series (2.34) and we seek a solution u of

$$-\Delta u + u = f \quad \text{on} \quad \mathbb{T}^n \tag{5.11}$$

having a Fourier series (2.35), then from (2.36), matching the coefficients we find that

$$u_k = \frac{f_k}{1 + |k|^2}$$

so

$$u(x) = \sum_k \frac{f_k}{1 + |k|^2} e^{ik\cdot x}. \tag{5.12}$$

Moreover, if $f \in H^s(\mathbb{T}^n)$ then

$$\|\varphi\|^2_{H^{s+2}(\mathbb{T}^n)} = \sum_k (1+|k|^2)^{s+2}\frac{|f_k|^2}{(1 + |k|^2)^2} = \sum_k (1+|k|^2)^s |f_k|^2 = \|f\|^2_{H^s(\mathbb{T}^n)}.$$

We summarize this.

Theorem 5.1. *Given any $f \in H^s(\mathbb{T}^n)$, there is a unique solution u of $-\Delta u + u = f$. Moreover, u is in $H^{s+2}(\mathbb{T}^n)$, that is, it has two more derivatives than f in L^2. If $s > j + n/2$, then, by the Sobolev embedding theorem 2.9, $u \in C^j(\mathbb{T}^2)$.*

So far we only considered the case where $f \in H^s(\mathbb{T}^n)$. This is a global assumption on the smoothness of f. What can one say if f happens to be smoother only near a point x_0? We suspect that the smoothness of u near x_0 will depend only on the smoothness of f near x_0. This is easy.
Say we know that F is smoother in the ball $B(x_0, R)$. Pick an $r < R$ and a non-negative smooth function $\eta(x)$ so that $\eta(x) > 0$ in the ball $B(x_0, R)$ with

$$\eta(x) = \begin{cases} 1 & \text{for } x \text{ in } B(x_0, r), \\ 0 & \text{for } x \text{ outside } B(x_0, R). \end{cases}.$$

Extend $f(x)$ to $\mathbb{T}^n$ by $f(x) = 0$ outside $B(x_0, R)$. Our vague smoothness assumption on f near x_0 is now made precise by formally assuming that $(\eta f) \in H^s(\mathbb{T}^n)$ for all $r < R$ (one can use this to define the spaces H^s_{loc} of functions that are locally in H^s).
If $u \in H^2(\mathbb{T}^n)$ satisfies $-\Delta u + u = f$, consider $v(x) := \eta(x)u(x)$ and use the obvious extensions to $\mathbb{T}^n$. Then,

$$-\Delta v + v = -(\eta\Delta u + 2\nabla\eta \cdot \nabla u + u\Delta\eta) + \eta u = G,$$

where

$$G = \eta f - 2\nabla\eta \cdot \nabla u - u\Delta\eta. \tag{5.13}$$

Say $f \in H^s(\mathbb{T}^n)$ for some $s \geq 1$. Since $u \in H^2(\mathbb{T}^n)$, then G is in $H^1(\mathbb{T}^n)$. Thus by Theorem 5.1, $v \in H^3(\mathbb{T}^n)$. If $s \geq 2$ we can repeat this to find that $G \in H^2(\mathbb{T}^n)$ so $v \in H^3(\mathbb{T}^n)$. Continuing, we conclude that $f \in H^s(\mathbb{T}^n)$ implies $v \in H^{s+2}(\mathbb{T}^n)$. But $u = v$ in $B(x_0, r)$ so u is in H^s in this ball. This proves

COROLLARY 5.2. [LOCAL REGULARITY] *If $u \in H^2(\mathbb{T}^n)$ satisfies $-\Delta u + u = f$ and $f \in H^s$ in a neighborhood of x_0, then $u \in H^{s+2}$ in this neighborhood.*

Exercise: If $f \in H^s(\mathbb{T}^n)$, discuss the existence, uniqueness, and regularity of solutions to $-\Delta u = f$ on $\mathbb{T}^n$.

Exercise: Use Duhamel's Principle to find a simple formula for the solution of $-u'' + u = f(x)$ for $0 < x < \pi$, with $u(0) = u(\pi) = 0$. Compare this with the solution obtained using Fourier series.

6. Harmonic polynomials and spherical harmonics

Consider the linear space $\mathcal{P}_\ell$ of polynomials of degree at most ℓ in the n variables $x_1, \ldots, x_n$ and let $\mathcal{P}_\ell$ be the sub-space of polynomials homogeneous of degree ℓ. A polynomial $u(x)$ is called a *harmonic polynomial* if $\Delta u = 0$. We wish to compute the dimension of the subspace H_ℓ of $\mathcal{P}_\ell$ consisting of harmonic polynomials, homogeneous of degree ℓ. If $n = 2$, and $\ell \geq 1$ the dimension is 2, since for $\ell \geq 1$ one basis for the space of harmonic polynomials of degree exactly ℓ is the real and imaginary parts of the analytic function $(x + iy)^\ell$.
For the general case, observe that $\Delta : \mathcal{P}_{\ell+2} \to \mathcal{P}_\ell$ and define the linear map $L : \mathcal{P}_\ell \to \mathcal{P}_\ell$ by the formula

$$Lp(x) := \Delta\big[(|\,x\,|^2 - 1)p(x)\big], \tag{5.14}$$

where $|\,x\,|$ is the Euclidean norm. Now $Lp = 0$ means the polynomial $u(x) := (|\,x\,|^2 - 1)p(x) \in \mathcal{P}_{\ell+2}$ is harmonic. But clearly $u(x) = 0$ on the sphere $|\,x\,| = 1$, so $u \equiv 0$. Thus $\ker L = 0$ so L is invertible. In particular, given a homogeneous $q \in \mathcal{P}_\ell$ there is a $p \in \mathcal{P}_\ell$ with $\Delta\big[(|\,x\,|^2 - 1)p(x)\big] = q$. Let $v \in \mathcal{P}_\ell$ denote the homogeneous part of p that has highest degree ℓ. Since Δ reduces the degree by two, we deduce that in fact $\Delta(|\,x\,|^2 v) = q$. Therefore this map $v \mapsto q$ from $\mathcal{P}_\ell \to \mathcal{P}_\ell$ is onto and hence an isomorphism.[1] Here are two consequences.

1) Since the map $\Delta : \mathcal{P}_\ell \to \mathcal{P}_{\ell-2}$ is onto, again by linear algebra, we can compute the dimension of the space of homogeneous harmonic polynomials:

$$\dim H_\ell = \dim \mathcal{P}_\ell - \dim \mathcal{P}_{\ell-2} = \binom{n+\ell-1}{\ell} - \binom{n+\ell-3}{\ell-2} = \frac{(n+2\ell-2)(n+\ell-3)!}{\ell!(n-2)!}.$$

For instance if $n = 3$ then $\dim H_\ell = 2\ell + 1$.

2) Any homogeneous polynomial $q \in \mathcal{P}_\ell$ can be written (uniquely) in the form $q = h + |\,x\,|^2 v$, where $h \in H_\ell$ and $v \in \mathcal{P}_{\ell-2}$. To prove this, first compute Δq and then use the above to find a unique $v \in \mathcal{P}_{\ell-2}$ so that $\Delta(|\,x\,|^2 v) = \Delta q \in \mathcal{P}_{\ell-2}$. The function $h := q - |\,x\,|^2 v$ is clearly harmonic. Applying this again to v and so on recursively we conclude that $q = h_\ell + |\,x\,|^2 h_{\ell-2} + |\,x\,|^4 h_{\ell-4} + \cdots$, where $h_j \in H_j$. This yields the direct sum decomposition $\mathcal{P}_\ell = H_\ell \oplus |\,x\,|^2 H_{\ell-2} \oplus \cdots$. Since both the Laplacian and the operation of multiplying by $|\,x\,|^2$ commute with rotations, the summands in this decomposition are $SO(n)$-invariant, a fact that is useful in discussing spherical harmonics and the symmetry group $SO(n)$.

The idea behind the definition of L in (5.14) was that to solve $\Delta u = q \in \mathcal{P}_\ell$, we seek u in the special form $u = (|\,x\,|^2 - 1)p(x)$ to obtain a new problem, $Lp = q$, whose solution is unique. Frequently it is easier to solve a problem if you restrict the form of the solution to obtain uniqueness.

Homogeneous harmonic polynomials arise since, when restricted to the unit sphere these are exactly the eigenfunctions of the Laplacian on the sphere. These are called the *spherical harmonics*. The dimensions of the eigenspaces are then the numbers just computed. for instance, when $n = 3$ this number is $2\ell + 1$. We carry our part of this computation.
In spherical coordinates on $\mathbb{R}^n$, the Laplacian is

$$\Delta_{\mathbb{R}^n} u = \frac{\partial^2 u}{\partial r^2} + \frac{n-1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\Delta_{S^{n-1}} u, \tag{5.15}$$

where $\Delta_{S^{n-1}}$ is the Laplacian on the standard sphere S^{n-1}. If $p(x)$ is a polynomial, homogeneous of degree k, then $p(x) = r^k v(\xi)$, where ξ is a point on the unit sphere. Thus, if p is also harmonic, then using (5.15)

$$0 = \Delta_{\mathbb{R}^n} r^k v(\xi) = r^{k-2}[(k(k-1) + (n-1)k)v + \Delta_{S^{n-1}}]v.$$

Consequently,

$$-\Delta_{S^{n-1}} v = k(n+k-2)v.$$

In other words, v is an eigenfunction of the Laplacian on S^{n-1} and the corresponding eigenvalue is $k(n+k-2)$. The missing piece is to show that every eigenfunction of the Laplacian has this form. This can be done, for instance, by proving that the dimension of the eigenspace is the same as the dimension of the space of homogeneous harmonic polynomials.

[1] One can also give a purely algebraic proof that if $p \in \mathcal{P}_\ell$ satisfies $\Delta(|\,x\,|^2 p) = 0$, then $p \equiv 0$ —hence the map $M : \mathcal{P}_\ell \mapsto \mathcal{P}_\ell$ defined by $Mp := \Delta(|\,x\,|^2 p)$ is an isomorphism of $\mathcal{P}_\ell$.

APPLICATION. Atoms are roughly spherically symmetric. The maximum number of electrons in the k^{th} atomic subshell is related to the dimension of the eigenspace corresponding to the k^{th} eigenvalue. The Pauli exclusion principle asserts that no two electrons can be in the same state. But electrons can have spins $\pm 1/2$, There are $2k+1$ electrons with spin $\pm\frac{1}{2}$, so $2(2k+1)$ in all. Thus the subshells contain at most $2, 6, 10, 14, \dots$ electrons.

7. Dirichlet's principle and existence of a solution

a). History. To solve the Dirichlet problem (5.9), Dirichlet proposed to find the function u that minimizes the *Dirichlet integral*

$$J(\varphi) := \int_\Omega |\,\nabla\varphi\,|^2\,dx \tag{5.16}$$

among all functions φ, say piecewise smooth, with $\varphi = f$ on $\partial\Omega$. To see this, let h be any function that is piecewise smooth in Ω and zero on the boundary. If u minimizes $J(\varphi)$, then for any t the function $\varphi = u+th$ has the correct boundary values so $J(u+th)$ has a minimum at $t = 0$. Taking the first derivative gives

$$0 = \frac{dJ(u+th)}{dt}\Big|_{t=0} = 2\int_\Omega \nabla u\cdot\nabla h\,dx. \tag{5.17}$$

If u has two continuous derivatives, we can now integrate by parts and use that $h = 0$ on the boundary to find

$$0 = \int_\Omega (\Delta u)\,h\,dx.$$

Since h can be any piecewise smooth function that is zero on the boundary, this implies that $\Delta u = 0$, as desired. [PROOF: if not, say $\Delta u > 0$ somewhere, then $\Delta u > 0$ on a small ball. Pick a function h that is positive on this ball and zero elsewhere, giving the contradiction $\int_\Omega \Delta u\,h\,dx > 0$.]

Riemann adopted this reasoning in his proof of what we now call the Riemann mapping theorem. Weierstrass pointed out that although $J(u)$ is bounded below and hence has an infimum, it is not evident that that there is some function u satisfying the boundary conditions for which $J(u)$ has that minimal value. To make his argument convincing, he gave the example

$$J(\varphi) = \int_{-1}^{1} x^2\varphi'(x)^2\,dx \qquad \text{with } \varphi(\pm 1) = \pm 1.$$

For this consider the sequence

$$\varphi_k(x) = \begin{cases} -1 & \text{for } -1 \le x \le -1/k, \\ kx & \text{for } -1/k \le x \le 1/k, \\ 1 & \text{for } \ 1/k \le x \le 1. \end{cases}$$

Then $J(\varphi_k) = \frac{2}{3k} \to 0$, so $\inf J(\varphi) = 0$. But if $J(u) = 0$, then $u = \text{const}$ and can't satisfy the boundary conditions.

Since Riemann's application of Dirichlet's principle was important, many people worked on understanding the issues. Using other methods Poincaré gave a rather general proof that one could solve the Dirichlet problem (5.9) while around 1900 Hilbert showed that under reasonable conditions, Dirichlet's principle is indeed valid.

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b). A modified problem. In subsequent years the tools developed to understand the issues have led to a considerable simplification. First, instead of solving (5.9) solve the related inhomogeneous equation

$$-\Delta u = F \ \text{ in } \ \Omega \qquad \text{with} \qquad u = 0 \text{ on } \partial\Omega. \tag{5.18}$$

To reduce (5.9) to this form, let $f_e(x)$ be a smooth extension of f from $\partial\Omega$ to all of Ω. We assume this can be done since if there is a solution of (5.9), then the solution itself gives a very special extension. Then let $w := u - f_e$. This satisfies $-\Delta w = \Delta f_e$, which has the form (5.18) with $F = \Delta f_e$.

For (5.18) the analogue of (5.16) is the functional

$$Q(\varphi) := \int_\Omega \left[|\,\nabla\varphi\,|^2 - 2F\varphi\right]\,dx.$$

Imitating the procedure Dirichlet followed, we seek to minimize Q from an appropriate class of functions that vanish on the boundary. If u minimizes Q, then, $Q(u+th)$ has its minimum at $t = 0$. Computing $dQ/dt|_{t=0}$ gives

$$\int_\Omega (\nabla u\cdot\nabla h - Fh)\,dx = 0 \tag{5.19}$$

for all h that vanish on the boundary. As before, assuming this function u has two continuous derivatives, an integration by parts shows that $-\Delta u = F$, as desired. It is not difficult to show that Q is bounded below, but even knowing this we still don't know that Q achieves its minimum. Instead of perusing this, we take a slightly different approach.

For a bounded open set Ω, use the space $C_c^1(\Omega)$ of functions with compact support in Ω (the *support* of a function is the closure of the set where the function is not zero). Thus, the functions in $C_c^1(\Omega)$ are zero near the boundary of Ω. For $\varphi \in C_c^1(\Omega)$ define the norm

$$\|\varphi\|^2_{H_0^1(\Omega)} = \int_\Omega |\,\nabla\varphi\,|^2\,dx.$$

Because of the Poincaré inequality (3.34), this is a norm, not a seminorm. Define the *Sobolev space* $H_0^1(\Omega)$ as the completion of $C_c^1(\Omega)$ in this norm. This is a Hilbert space with inner product

$$\langle \varphi, \psi \rangle_{H_0^1(\Omega)} = \int_\Omega \nabla\varphi \cdot \nabla\psi \, dx.$$

Motivated by (5.19) If $F \in L_2(\Omega)$, we say $u \in H_0^1(\Omega)$ is a *weak solution* of (5.18) if

(5.20)
$$\int_\Omega \nabla u \cdot \nabla v \, dx = \int_\Omega F v \, dx, \qquad \text{that is} \qquad \langle u, v \rangle_{H_0^1(\Omega)} = \int_\Omega f v \, dx$$

for all $v \in H_0^1(\Omega)$. If $u \in \mathbb{C}^2(\Omega)$ satisfies (5.18), then it is clearly a weak solution: just integrate by parts. Conversely, if $u \in \mathbb{C}^2(\Omega)$ is a weak solution, then after an integration by parts,

$$\int_\Omega [-\Delta u - F] v \, dx = 0 \qquad \text{for all} \quad v \in H_0^1(\Omega),$$

so, arguing as above, $-\Delta u = F$. By a separate argument that we do not give, $u = 0$ on $\partial\Omega$.

Note that a weak solution, if one exists, is unique, since if there were two, u and w let $\varphi = u - w \in H_0^1(\Omega)$. Then

$$\int_\Omega \nabla w \cdot \nabla v \, dx = 0 \qquad \text{for all} \quad v \in H_0^1(\Omega).$$

Letting $v = w$ we conclude that $\int_\Omega |\nabla w|^2 = 0$ so, using the Poincare inequality (3.34), $w = 0$. Consequently, if we have a weak solution and if we believe there is a classical solution, then the only possibility is that the weak solution is also the desired classical solution.

Our strategy is to break the proof of the existence of a solution into two parts:

Existence: Prove there is a weak solution.
Regularity: Prove that this weak solution is a classical solution – if f is smooth enough.

c). Existence of a weak solution. The key ingredient in the following proof of the existence of a weak solution is a standard result in elementary functional analysis: the *Riesz representation theorem* for a Hilbert space $\mathcal{H}$. To state it, recall that a bounded linear functional $\ell(x)$ is a linear map from elements $x \in \mathcal{H}$ to the complex numbers with the property that $|\ell(x)| \le c\|x\|$, where the real number c is independent of x. A simple example is $\ell(x) = \langle x, z \rangle$ for some $z \in \mathcal{H}$. The Riesz representation theorem states that every bounded linear functional has this form.

For those new to this result, here is a primitive proof (using coordinates) that works for separable Hilbert spaces. As a warm-up, first in finite

dimensions. In an orthonormal basis $e_1, \dots e_n$, say $x = x_1 e_1 + \cdots x_n e_n$. Then by linearity

$$\ell(x) = x_1 \ell(e_1) + \cdots + x_n \ell(e_n).$$

Consequently, if we let $z = \overline{\ell(x_1)} e_1 + \cdots + \overline{\ell(e_n)} e_n$, then $\ell(x) = \langle x, z \rangle$. Geometrically one can interpret z as a vector orthogonal to the kernel of ℓ.

Essentially the same proof works in any separable Hilbert space. Pick a (countable) orthonormal basis and write $x \in \mathcal{H}$ in this basis. Then, as above, we are led to let

$$z = \overline{\ell(x_1)} e_1 + \cdots + \overline{\ell(e_n)} e_n + \cdots .$$

However, it is not yet evident that this series converges in $\mathcal{H}$. Thus, for any N let

$$z_N = \overline{\ell(e_1)} e_1 + \cdots + \overline{\ell(e_N)} e_N.$$

Then,

$$|\ell(z_N)| = \overline{\ell(e_1)}\ell(e_1) + \cdots + \overline{\ell(e_N)}\ell(e_N) = \|z_N\|^2.$$

But by hypothesis, $|\ell(x)| \le c\|x\|$, so $\|z_N\|^2 \le c\|z_N\|$. Thus $\|z_N\| \le c$. Because this estimate is independent of N, the series defining z_N converges in $\mathcal{H}$ to an element of $\mathcal{H}$ and we have $\ell(x) = \langle x, z \rangle$ for all $x \in \mathcal{H}$.

To prove the existence of a weak solution, motivated by (5.20), for any $v \in H_0^1(\Omega)$, define the linear functional

$$\ell(v) := \int_\Omega F v \, dx.$$

Then by the Schwarz and Poincare inequalities

$$|\ell(v)| = \|F\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} \le c\|v\|_{L^2(\Omega)} \le c\|v\|_{H_0^1(\Omega)},$$

By the Riesz representation theorem, there is a $u \in H_0^1(\Omega)$ such that

$$\ell(v) = \langle u, v \rangle_{H_0^1(\Omega)}, \quad \text{that is,} \quad \langle u, v \rangle_{H_0^1(\Omega)} = \int_\Omega F v \, dx,$$

just as desired.

Note that this proof works for any bounded open set Ω, no matter how wild its boundary. For instance, if Ω is the punctured sphere $0 < \|x\| < 1$ in $\mathbb{R}^3$ and try to solve $-\Delta u = 4$ there with $u = 0$ on the boundary, the unique solution in polar coordinates is $u = (1 - |r|^2)$ which does not satisfy the boundary condition we attempted to impose, $u(0) = 0$. That jump discontinuity is a removable singularity. The existence theorem is smart enough to ignore bad points we may have on the boundary of Ω.

Exercises:

1. In the one dimensional case, so $\Omega \subset \mathbb{R}^1$ is a bounded interval, if $u \in H_0^1(\Omega)$, show that $u \in C(\bar{\Omega})$ and $u = 0$ on $\partial\Omega$.

2. Let $\Omega \in \mathbb{R}^1$ be a bounded open interval, $a(x) \in C^1(\bar{\Omega}$ satisfy $0 < \alpha \le a(x) \le \beta$ and $c(x) \ge 0$ a bounded continuous function. Consider solving
$$Lu := -(a(x)u')' + c(x)u = f \in L^2(\Omega) \qquad \text{with} \quad u = 0 \quad \text{for} \quad x \in \partial\Omega.$$
Define $u \in H_0^1(\Omega)$ to be a *weak solution* of $Lu = f$ if
$$\int_\Omega [a(x)u'v' + c(x)uv]\,dx = \int_\Omega fv\,dx$$
for all $v \in H_0^1(\Omega)$. Prove that there exists exactly one weak solution. [SUGGESTION: Define and use a Hilbert space that uses $\int_\Omega [a(x)\varphi'\psi' + c\varphi\psi]\,dx$ as its inner product. Show that the norm on this space is equivalent to the $H_0^1(\Omega)$ norm.]

3. If $c(x) > 0$ is a continuous function in $\bar{\Omega}$ and $F \in L^2(\Omega)$, prove there is a unique weak solution $u \in H_0^1(\Omega)$ of $-\Delta u + c(x)u = F$. [The first step is to define a "weak solution"].

4. Let $A := (a_{ij}(x))$ be a positive definite $n \times n$ matrix of continuously differentiable functions for $x \in \bar{\Omega}$, where $\Omega \subset \mathbb{R}^n$ is a bounded open set. In particular, there are constants m, M so that for any vector $v \in \mathbb{R}^n$ we have $m\|v\|^2 \le \langle v,\, Av\rangle \le M\|v\|^2$. Consider
$$Lu := -\sum_{i,j=1}^n \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right) + c(x)u = F(x)$$
where $0 \le c(x) \le \gamma$ is continuous and $F \in L^2(\Omega)$. Show there is a unique weak solution $u \in H_0^1(\Omega)$ of $Lu = F \in L^2(\Omega)$. [The first step is to define a "weak solution"].

d). Regularity of the weak solution. If needed, dilate our bounded domain Ω, so it is inside the box $|x_j| \le \pi$, $j = 1, \ldots, n$, which we view as the torus $\mathbb{T}^n$. Let $u \in H_0^1(\Omega)$ be our weak solution of $-\Delta u = F \in L^2(\mathbb{T}^n)$. Rewrite this as $-\Delta u + u = f(x)$, where now $f(x) = F(x) + u(x) \in L^2(\mathbb{T}^n)$ is considered to be a known function. By Theorem 5.1, $u \in H^2(\mathbb{T}^2)$.
Although F might be smoother in Ω, our extension of F to $\mathbb{T}^n$ likely looses this additional smoothness across $\partial\Omega$. However, the local regularity Corollary 5.2 implies that if F is in H^s near a point x_0, then u is in H^{s+2} near x_0.

CHAPTER 6

The Rest

In the last part of the course I outlined the several topics, mainly following various parts from my old notes
Lecture Notes on Applications of Partial Differential Equations to Some Problems in Differential Geometry, available at

`http://www.math.upenn.edu/~kazdan/japan/japan.pdf`

In addition, there is a bit of overlap with my expository article *Solving Equations* available at
`http://www.math.upenn.edu/~kazdan/solving/solvingL11pt.pdf`

Topics

- Defined both the Hölder spaces $C^{k+\alpha}$, $0 \leq \alpha \leq 1$ and Sobolev spaces $H^{p,k}$ and illustrated how to use them in various regularity assertions for solutions of some linear and nonlinear elliptic partial differential equations.
- Defined ellipticity for nonlinear equations, giving several examples including a Monge-Ampère equation.
- Discussed issues concerning qualitative properties and existence for the minimal surface equation, equations of prescribed mean and Gauss curvature (for surfaces) and some equations for steady inviscid fluid flow.
- Discussed techniques for proving that a partial differential equation has a solution. The techniques included:
  a) iteration using contracting mappings,
  b) direct methods in the calculus of variations,
  c) continuity method
  d) fixed point theorems (Schauder and Leray),
  e) heat equation (R. Hamilton).