Math 313/515, Spring 2005 Jerry L. Kazdan

Homework Set 8, Due Thursday, March 24, 2005

(*Late papers will be accepted until 4 PM on Fri. March 25*)

To get experience with simple calculations, where specified do the calculations by hand, not with a calculator or computer.

1. Let A be a 5×5 matrix whose *second* column is $V=(0,-3,0,0,0)$. Show that V is an eigenvector of A and find the corresponding eigenvalue.

2. Let $S=\begin{pmatrix}5 & 7\\ 2 & 3\end{pmatrix}$, $\quad D=\begin{pmatrix}-2 & 0\\ 0 & 1\end{pmatrix}$, and $B=SDS^{-1}$.

   a) Compute B^4 (by hand).

   b) What are the eigenvalues and corresponding eigenvectors of B?

   c) What are the eigenvalues and corresponding eigenvectors of $C:=B+3I$?

3. Diagonalize (by hand): $\begin{pmatrix}3 & 0 & 0\\ 0 & 3 & 0\\ 1 & 4 & -2\end{pmatrix}$.

4. By hand, find the eigenvalues and corresponding eigenvectors of the following matrices. [Complex numbers might be needed.]

   a). $A=\begin{pmatrix}3 & 1\\ 1 & 3\end{pmatrix}$ b). $B=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$ c). $C=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$ d). $D=\begin{pmatrix}3 & 0\\ 3 & 3\end{pmatrix}$.

5. Let the matrix $A=(a_{ij})$ have the special form $a_{ij}=b_ic_j$, where $b=(2,0,-1,1)$ and $c=(1,2,-1,1)$.

   a) Compute the rank of A.

   b) Find a basis for the nullspace of A.

   c) Compute the trace of A.

   d) Find the eigenvalues and corresponding eigenvectors.

   e) Find a matrix S so that $S^{-1}AS$ is a diagonal matrix.

   f) Use this to compute A^{100}.

6. Let T be a 6×6 matrix with rank 1 and $\text{trace}(T)=5$.
   a) What is $\det T$? Why?
   b) What are all the eigenvalues of T?
   c) Can any such T be diagonalized? Why?
   d) What are all the eigenvalues of $B := T-2I$?
   e) $\det B = ?$

7. Let $Z := (z_1, z_2, \ldots, z_n) \neq 0$, b, and c be given real numbers, $A := (z_i z_j)$, so A has rank one, and let $M = bA + cI$. What are the eigenvalues of M? [The answer of course involves a, b, and Z.]

8. Let M be an anti-symmetric 5×5 matrix (so $M^T = -M$). Show that $\det M = 0$.

9. Describe all matrices C with the property $\det C + \det C = \det(C+C)$.

10. Let A be a 5×5 matrix with eigenvectors V_1, ..., V_5 and corresponding eigenvalues λ_1, ..., λ_5. Say $X = 2V_1 - V_2 + V_4 + 3V_5$.
    a) Compute AX, A^2X, and A^4X in terms of the eigenvalues and eigenvectors.
    b) If $\lambda_1 = 1$ and that the remaining eigenvalues have absolute value less than one: $|\lambda_j| < 1,\ j = 2,3,4,5$, compute $\lim_{k\to\infty} A^k X$.