Math 312, Spring 2013 Jerry L. Kazdan

Problem Set 9

DUE: In class Thursday, Apr. 11. *Late papers will be accepted until 1:00 PM Friday.*

Lots of problems. Fortunately many are short.

1. This asks you to come up with four examples. In each case, find a real matrix (perhaps 2×2) that is:
   a) Both invertible and diagonalizable.
   b) Not invertible, but diagonalizable.
   c) Not diagonalizable but is invertible.
   d) Neither invertible nor diagonalizable.

2. Let $A := \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix}$.
   a) Find the eigenvalues of A.
   b) Is the origin a stable equilibrium of the discrete dynamical system $\vec{x}_{k+1} = A\vec{x}_k$? Explain.

3. [BRETSCHER, SEC. 7.5 #14] Let $A = \begin{pmatrix} 1 & -2 \\ 1 & -1 \end{pmatrix}$. Find an invertible matrix S so that $S^{-1}AS = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$.

4. [BRETSCHER, SEC. 7.6 #18] If $\vec{x}(t+1) = A\vec{x}(t)$, where $A := \begin{pmatrix} -0.8 & 0.6 \\ -0/8 & -0.8 \end{pmatrix}$ and $\vec{x}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, find a real closed formula for the trajectory $\vec{x}(t)$. Also, draw a rough sketch.

5. [BRETSCHER, SEC. 7.5 #24] Find all the eigenvalues of $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 5 & -7 & 3 \end{pmatrix}$.

6. [BRETSCHER, 5th ed SEC. 7.5 #32(A)] Consider the dynamical system $\vec{x}(t+1) = A\vec{x}(t)$, where $A := \begin{pmatrix} 0.4 & 0.1 & 0.5 \\ 0.4 & 0.3 & 0.1 \\ 0.2 & 0.6 & 0.4 \end{pmatrix}$, perhaps modeling the way people search a mini-web. Using technology (say the Maple example I did in class:

   `http://hans.math.upenn.edu/~kazdan/312S13/Maple/MarkovChain.mw`),

   compute high powers of A, say A^6, A^{16} and A^{32}), and make a conjecture about $\lim_{t\to\infty} A^t$.

7. [Bretscher, Sec. 7.3 #28] Let $B := \begin{pmatrix} k & 1 & 0 & 0 \\ 0 & k & 1 & 0 \\ 0 & 0 & k & 1 \\ 0 & 0 & 0 & k \end{pmatrix}$ where k is an arbitrary constant. Find the eigenvalue(s) of B and determine both their algebraic and geometric multiplicities. [Note: First try the analogous 2×2 case.]

8. Let A be an $n \times n$ real matrix. If A is orthogonally similar to a real diagonal matrix D, must A be symmetric? Proof or counterexample [The matrices A and B are *orthogonally similar* if $A = RBR^{-1}$ for some orthogonal matrix R.]

9. [Bretscher, Sec. 8.1 #24] Find an orthonormal eigenbasis for $\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$.

10. [Bretscher, Sec. 8.1 #38] Let A be a symmetric 2×2 matrix with eigenvalues -2 and 3 and $u \in \mathbb{R}^2$ any unit vector. What are the possible values of $\langle u, Au \rangle$? Illustrate your answer in terms of the unit circle and its image under A.

11. Of the following three matrices, one can be orthogonally diagonalized; one can be diagonalized (but not orthogonally); and one cannot be diagonalized at all. Identify these – *fully explaining your reasoning.*

$$A = \begin{bmatrix} 0 & 3 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 3 & 1 \\ 3 & 0 & 2 \\ 1 & 2 & 0 \end{bmatrix}, \qquad C = \begin{bmatrix} 3 & 1 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}.$$

12. [Bretscher, Sec. 8.2 #18] Sketch the curve of points in the plane that satisfy $9x_1^2 - 4x_1x_2 + 6x_2^2 = 1$.

13. a) Let $D := \begin{pmatrix} 4 & 0 \\ 0 & 25 \end{pmatrix}$ Find a positive definite symmetric matrix P so that $P^2 = D$ (we call P the *square root* of D)

    b) Let $A := \begin{pmatrix} 10 & 6 \\ 6 & 10 \end{pmatrix}$. Find a positive definite (symmetric) matrix P so that $P^2 = A$.

    c) Show that every positive definite symmetric matrix A has a positive definite square root.

14. [Bretscher, Sec. 8.2 #28] Show that any positive definite $n \times n$ matrix A can be written as $A = BB^*$, where the columns of B are orthogonal. [Hint: Use the result of the previois problem.]

15. [Bretscher, Sec. 8.2 #26] Consider the quadratic polynomial $Q(\vec{x}) := \langle \vec{x}, A\vec{x} \rangle$, where A is a real $n \times n$ symmetric matrix. "If for some vector $\vec{v} \neq 0$ we know that $Q(\vec{v}) = 0$, then A cannot be invertible." Proof or counterexample.

16. Let $f(x, y) := (x^2 + 4y^2)e^{(1-x^2-y^2)}$. Find and classify all of its critical points as local maxima etc.

[Last revised: May 5, 2013]